ON KATO-SOBOLEV SPACES. THE WIENER-LÉVY THEOREM FOR KATO-SOBOLEV ALGEBRAS $\mathcal{H}^s_{\mathtt{ul}}$.

GRUIA ARSU

Abstract. We investigate some multiplication properties of Kato-Sobolev spaces by adapting the techniques used in the study of Beurling algebras by Coifman and Meyer [Co-Me]. Also we develop an analytic functional calculus for Kato-Sobolev algebras based on an integral representation formula belonging A. P. Calderón.

Contents

1. Introduction

In this paper we study some multiplication properties of Kato-Sobolev spaces and we develop an analytic functional calculus for Kato-Sobolev algebras. Kato-Sobolev spaces $\mathcal{H}^s_{\mathtt{ul}}$ were introduced in [K] by Tosio Kato and are known as uniformly local Sobolev spaces. The uniformly local Sobolev spaces can be seen as a convenient class of functions with the local Sobolev property and certain boundedness at infinity. We mention that $\mathcal{H}^s_{\mathtt{ul}}$ were defined only for integers $s \geq 0$ and play an essential part in the paper. In this paper, Kato-Sobolev spaces are defined for arbitrary orders and are proved some embedding theorems (in the spirit of the [K]) which expresses the multiplication properties of the Kato-Sobolev spaces. The techniques we use in establishing these results are inspired by techniques used in the study of Beurling algebras by Coifman and Meyer [Co-Me]. Also we develop an analytic functional calculus for Kato-Sobolev algebras based on an integral representation formula of A. P. Calderón. This part corresponds to the section of [K] where the invertible elements of the algebra $\mathcal{H}^s_{\mathtt{ul}}$ are determined and which has as main result a Wiener type lemma for $\mathcal{H}^s_{\mathtt{ul}}$. In our case, the main result is the Wiener-Lévy theorem for Kato-Sobolev algebras. This theorem allows a spectral analysis of these algebras. In Section 2 we define Sobolev spaces of multiple order. Uniformly local Sobolev spaces of multiple order were used as spaces of symbols of pseudo-differential operators in many papers [B1], [B2],.... By adapting the techniques of Coifman and Meyer, used

2000 *Mathematics Subject Classification.* Primary 46E35, 46E25, 46Jxx; Secondary 46H30, 42B35.

in the study of Beurling algebras, we prove a result that allows us to extend the embedding theorems of Kato in the case when the order of $\mathcal{H}_{\mathtt{ul}}^{s}$ is not an integer ≥ 0. In Section 3 we study an increasing family of spaces $\left\{\mathcal{K}_{p}^{\mathbf{s}}\right\}_{1\leq p\leq\infty}$ for which $\mathcal{K}_{\infty}^{\mathbf{s}}=\mathcal{H}_{\mathtt{ul}}^{\mathbf{s}}$. The Wiener-Lévy theorem for Kato-Sobolev algebras is established in Section 4. Using this theorem we build an analytic functional calculus for Kato-Sobolev algebras.

2. Sobolev spaces of multiple order

Let $j\in\{1,...,n\}$. Suppose that $\mathbb{R}^{n}=\mathbb{R}^{n_{1}}\times...\times\mathbb{R}^{n_{j}}$, where $n_{1},...,n_{j}\in\mathbb{N}^{*}$. We have a partition of variables corresponding to this orthogonal decomposition, $\{1,...,n\}=\bigcup_{l=1}^{j}N_{l}$, where $N_{l}=\{k:n_{0}+...+n_{l-1}<k\leq n_{0}+...+n_{l}\}$. Here $n_{0}=0$ such that $N_{1}=\{1,...,n_{1}\}$.

Let $\mathbf{s}=(s_{1},...,s_{j})\in\mathbb{R}^{j}$. We find it convenient to introduce the following space

$$\mathcal{H}^{\mathbf{s}}\left(\mathbb{R}^{n}\right)=\left\{u\in\mathcal{S}'\left(\mathbb{R}^{n}\right):\left(1-\triangle_{\mathbb{R}^{n_{1}}}\right)^{s_{1}/2}\otimes...\otimes\left(1-\triangle_{\mathbb{R}^{n_{j}}}\right)^{s_{j}/2}u\in L^{2}\left(\mathbb{R}^{n}\right)\right\},$$
$$\left\|u\right\|_{\mathcal{H}^{\mathbf{s}}}=\left\|\left(1-\triangle_{\mathbb{R}^{n_{1}}}\right)^{s_{1}/2}\otimes...\otimes\left(1-\triangle_{\mathbb{R}^{n_{j}}}\right)^{s_{j}/2}u\right\|_{L^{2}},\quad u\in\mathcal{H}^{\mathbf{s}}.$$

For $\mathbf{s}=(s_{1},...,s_{k})\in\mathbb{R}^{k}$ we define the function

$$\langle\langle\cdot\rangle\rangle^{\mathbf{s}}:\mathbb{R}^{n}=\mathbb{R}^{n_{1}}\times...\times\mathbb{R}^{n_{k}}\to\mathbb{R},$$
$$\langle\langle\cdot\rangle\rangle^{\mathbf{s}}=\langle\cdot\rangle_{\mathbb{R}^{n_{1}}}^{s_{1}}\otimes...\otimes\langle\cdot\rangle_{\mathbb{R}^{n_{k}}}^{s_{k}},$$

where $\langle\cdot\rangle_{\mathbb{R}^{m}}=\left(1+\left|\cdot\right|_{\mathbb{R}^{m}}^{2}\right)^{1/2}$.Then

$$\langle\langle D\rangle\rangle^{\mathbf{s}}=\left(1-\triangle_{\mathbb{R}^{n_{1}}}\right)^{s_{1}/2}\otimes...\otimes\left(1-\triangle_{\mathbb{R}^{n_{k}}}\right)^{s_{k}/2},$$

and

$$\mathcal{H}^{\mathbf{s}}=\left\{u\in\mathcal{S}'\left(\mathbb{R}^{n}\right):\langle\langle D\rangle\rangle^{\mathbf{s}}u\in L^{2}\left(\mathbb{R}^{n}\right)\right\},$$
$$\left\|u\right\|_{\mathcal{H}^{\mathbf{s}}}=\left\|\langle\langle D\rangle\rangle^{\mathbf{s}}u\right\|_{L^{2}},\quad u\in\mathcal{H}^{\mathbf{s}}.$$

Let us note an immediate consequence of Peetre's inequality:

$$\langle\langle\xi+\eta\rangle\rangle^{\mathbf{s}}\leq2^{\left|\mathbf{s}\right|_{1}/2}\langle\langle\xi\rangle\rangle^{\mathbf{s}}\langle\langle\eta\rangle\rangle^{\left|\mathbf{s}\right|},\quad\xi,\eta\in\mathbb{R}^{n}$$

where $\left|\mathbf{s}\right|_{1}=\left|s_{1}\right|+...+\left|s_{k}\right|$ and $\left|\mathbf{s}\right|=\left(\left|s_{1}\right|,...,\left|s_{k}\right|\right)\in\mathbb{R}^{k}$. Also we have

$$\langle\langle\xi\rangle\rangle^{\mathbf{s}}\leq\langle\xi\rangle^{\left|\mathbf{s}\right|_{1}},\quad\xi\in\mathbb{R}^{n}.$$

Let k be an integer ≥0 or $k=\infty$. We shall use the following standard notations:

$$\mathcal{BC}^{k}\left(\mathbb{R}^{n}\right)=\left\{f\in\mathcal{C}^{k}\left(\mathbb{R}^{n}\right):f\text{ and its derivatives of order }\leq k\text{ are bounded}\right\},$$
$$\left\|f\right\|_{\mathcal{BC}^{l}}=\max_{m\leq l}\sup_{x\in X}\left\|f^{(m)}\left(x\right)\right\|<\infty,\quad l<k+1.$$

Proposition 2.1. *Suppose that* $\mathbb{R}^{n}=\mathbb{R}^{n_{1}}\times...\times\mathbb{R}^{n_{j}}$. *Let* $\mathbf{s}\in\mathbb{R}^{j}$, $k_{s}=\left[\left|\mathbf{s}\right|_{1}\right]+n+2$ *and* $m_{s}=\left[\left|\mathbf{s}\right|_{1}+\frac{n+1}{2}\right]+1$.

(a) $u\in\mathcal{H}^{\mathbf{s}}\left(\mathbb{R}^{n}\right)$ *if and only if there is* $l\in\{1,...,j\}$ *such that* $u,\partial_{k}u\in\mathcal{H}^{\mathbf{s}-\boldsymbol{\delta}_{l}}\left(\mathbb{R}^{n}\right)$ *for any* $k\in N_{l}$, *where* $\boldsymbol{\delta}_{l}=\left(\delta_{l1},...,\delta_{lj}\right)$.

($\mathtt{b}$) *If* $\chi \in H^{|\mathbf{s}|_1+\frac{n+1}{2}}(\mathbb{R}^n)$, *then for every* $u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$ *we have* $\chi u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$ *and*

$$\begin{aligned} \|\chi u\|_{\mathcal{H}^{\mathbf{s}}} &\leq C(s,n,\chi)\|u\|_{\mathcal{H}^{\mathbf{s}}} \\ &\leq C(s,n)\|\chi\|_{H^{|\mathbf{s}|_1+\frac{n+1}{2}}}\|u\|_{\mathcal{H}^{\mathbf{s}}}, \end{aligned}$$

where

$$\begin{aligned} C(s,n,\chi) &= (2\pi)^{-n} 2^{|\mathbf{s}|_1/2}\left(\int \langle\eta\rangle^{|\mathbf{s}|_1}|\widehat{\chi}(\eta)|\,\mathtt{d}\eta\right) \\ &\leq (2\pi)^{-n} 2^{|\mathbf{s}|_1/2}\left\|\langle\cdot\rangle^{-n-1}\right\|_{L^1}\|\chi\|_{H^{|\mathbf{s}|_1+\frac{n+1}{2}}} \\ &= C(s,n)\|\chi\|_{H^{|\mathbf{s}|_1+\frac{n+1}{2}}} \leq C(s,n)\left(\sum_{|\alpha|\leq m_s}\|\partial^\alpha\chi\|_{L^2}\right). \end{aligned}$$

Here $H^m(\mathbb{R}^n)$ *is the usual Sobolev space,* $m \in \mathbb{R}$.

($\mathtt{c}$) *If* $\chi \in \mathcal{C}^{k_s}(\mathbb{R}^n)$ *is* $\mathbb{Z}^n$*-periodic, then for every* $u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$ *we have* $\chi u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$.

($\mathtt{d}$) *If* $s_1 > n_1/2, ..., s_j > n_j/2$, *then* $\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{F}^{-1}L^1(\mathbb{R}^n) \subset \mathcal{C}_\infty(\mathbb{R}^n)$.

Proof. ($\mathtt{a}$) This part is trivial.

($\mathtt{b}$) Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{|\mathbf{s}|_1+\frac{n+1}{2}}(\mathbb{R}^n)$ (see the $\mathcal{B}_{p.k}$ spaces in Hörmander [Hö1] vol. 2), we can assume that $\chi, u \in \mathcal{S}(\mathbb{R}^n)$. In this case we have

$$\widehat{\chi u}(\xi) = (2\pi)^{-n}\widehat{\chi} * \widehat{u}(\xi).$$

Now we use Peetre's inequality and $\langle\langle\xi\rangle\rangle^{|\mathbf{s}|} \leq \langle\xi\rangle^{|\mathbf{s}|_1}$ to obtain

$$\langle\langle\xi\rangle\rangle^{\mathbf{s}}|\widehat{\chi u}(\xi)| \leq (2\pi)^{-n}2^{|\mathbf{s}|_1/2}\left(\int\langle\xi-\eta\rangle^{|\mathbf{s}|_1}|\widehat{\chi}(\xi-\eta)|\langle\langle\eta\rangle\rangle^{\mathbf{s}}|\widehat{u}(\eta)|\,\mathtt{d}\eta\right).$$

Then Schur's lemma implies that

$$\begin{aligned} \|\chi u\|_{\mathcal{H}^{\mathbf{s}}} &= \left\|\langle\langle\cdot\rangle\rangle^{\mathbf{s}}|\widehat{\chi u}|\right\|_{L^2} \\ &\leq (2\pi)^{-n}2^{|\mathbf{s}|_1/2}\left(\int\langle\eta\rangle^{|\mathbf{s}|_1}|\widehat{\chi}(\eta)|\,\mathtt{d}\eta\right)\left\|\langle\langle\cdot\rangle\rangle^{\mathbf{s}}|\widehat{u}|\right\|_{L^2} \\ &= C(s,n,\chi)\|u\|_{\mathcal{H}^{\mathbf{s}}} \end{aligned}$$

and Schwarz inequality gives the estimate of $C(s,n,\chi)$

$$\begin{aligned} C(s,n,\chi) &= (2\pi)^{-n} 2^{|\mathbf{s}|_1/2}\left(\int \langle\eta\rangle^{|\mathbf{s}|_1}|\widehat{\chi}(\eta)|\,\mathtt{d}\eta\right) \\ &\leq (2\pi)^{-n} 2^{|\mathbf{s}|_1/2}\left\|\langle\cdot\rangle^{-n-1}\right\|_{L^1}\|\chi\|_{H^{|\mathbf{s}|_1+\frac{n+1}{2}}} \\ &= C(s,n)\|\chi\|_{H^{|\mathbf{s}|_1+\frac{n+1}{2}}} \leq C(s,n)\left(\sum_{|\alpha|\leq m_s}\|\partial^\alpha\chi\|_{L^2}\right). \end{aligned}$$

($\mathtt{c}$) We shall use some results from [Hö1] vol. 1, pp 177-179, concerning periodic distributions. If $\chi \in \mathcal{C}^{k_s}(\mathbb{R}^n)$ is $\mathbb{Z}^n$-periodic, then

$$\chi = \sum_{\gamma\in\mathbb{Z}^n} \mathtt{e}^{2\pi\mathtt{i}\langle\cdot,\gamma\rangle}c_\gamma,$$

with Fourier coefficients

$$c_\gamma = \int_{\mathtt{I}} \chi(x)\, \mathtt{e}^{-2\pi \mathtt{i}\langle x,\gamma\rangle} \mathtt{d}x, \quad \mathtt{I} = [0,1)^n, \quad \gamma \in \mathbb{Z}^n,$$

satisfying

$$|c_\gamma| \le Cst\, \|\chi\|_{\mathcal{BC}^{k_s}(\mathbb{R}^n)} \langle 2\pi\gamma\rangle^{-k_s}, \quad \gamma \in \mathbb{Z}^n.$$

Since $\widehat{\mathtt{e}^{\mathtt{i}\langle\cdot,\eta\rangle} u} = \widehat{u}(\cdot - \eta)$, then Peetre's inequality implies that

$$\left\| \mathtt{e}^{\mathtt{i}\langle\cdot,\eta\rangle} u \right\|_{\mathcal{H}^{\mathbf{s}}} \le 2^{|\mathbf{s}|_1/2} \langle\langle\eta\rangle\rangle^{|\mathbf{s}|} \|u\|_{\mathcal{H}^{\mathbf{s}}} \le 2^{|\mathbf{s}|_1/2} \langle\eta\rangle^{|\mathbf{s}|_1} \|u\|_{\mathcal{H}^{\mathbf{s}}}.$$

It follows that

$$\begin{aligned} \|\chi u\|_{\mathcal{H}^{\mathbf{s}}} &\le Cst \cdot 2^{|\mathbf{s}|_1/2} \|\chi\|_{\mathcal{BC}^{k_s}(\mathbb{R}^n)} \left(\sum_{\gamma\in\mathbb{Z}^n} \langle 2\pi\gamma\rangle^{-k_s} \langle 2\pi\gamma\rangle^{|\mathbf{s}|_1} \right) \|u\|_{\mathcal{H}^{\mathbf{s}}} \\ &\le Cst \cdot 2^{|\mathbf{s}|_1/2} \left(\sum_{\gamma\in\mathbb{Z}^n} \langle 2\pi\gamma\rangle^{-n-1} \right) \|\chi\|_{\mathcal{BC}^{k_s}(\mathbb{R}^n)} \|u\|_{\mathcal{H}^{\mathbf{s}}}. \end{aligned}$$

(d) Let $u \in \mathcal{H}^{\mathbf{s}}$. If $s_1 > n_1/2, ..., s_j > n_j/2$, then $\widehat{u} \in L^1(\mathbb{R}^n)$ since $\langle\langle\cdot\rangle\rangle^{-\mathbf{s}}$, $\langle\langle\cdot\rangle\rangle^{\mathbf{s}} \widehat{u} \in L^2(\mathbb{R}^n)$. Now the Riemann-Lebesgue lemma implies the result. □

Lemma 2.2. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\theta \in [0,2\pi]^n$. If*

$$\varphi_\theta = \sum_{\gamma\in\mathbb{Z}^n} \mathtt{e}^{\mathtt{i}\langle\gamma,\theta\rangle} \varphi(\cdot - \gamma) = \sum_{\gamma\in\mathbb{Z}^n} \mathtt{e}^{\mathtt{i}\langle\gamma,\theta\rangle} \tau_\gamma \varphi,$$

then

$$\widehat{\varphi}_\theta = \nu_\theta = (2\pi)^n \sum_{\gamma\in\mathbb{Z}^n} \widehat{\varphi}(2\pi\gamma + \theta)\, \delta_{2\pi\gamma+\theta}.$$

Proof. We have

$$\varphi_\theta = \sum_{\gamma\in\mathbb{Z}^n} \mathtt{e}^{\mathtt{i}\langle\gamma,\theta\rangle} \varphi(\cdot - \gamma) = \sum_{\gamma\in\mathbb{Z}^n} \mathtt{e}^{\mathtt{i}\langle\gamma,\theta\rangle} \delta_\gamma * \varphi = \varphi * \left(\mathtt{e}^{\mathtt{i}\langle\cdot,\theta\rangle} S \right),$$

where $S = \sum_{\gamma\in\mathbb{Z}^n} \delta_\gamma$. We apply Poisson's summation formula, $\mathcal{F}\left(\sum_{\gamma\in\mathbb{Z}^n} \delta_\gamma\right) = (2\pi)^n \sum_{\gamma\in\mathbb{Z}^n} \delta_{2\pi\gamma}$, to obtain

$$\begin{aligned} \widehat{\varphi}_\theta &= \widehat{\varphi} \cdot \left(\widehat{\mathtt{e}^{\mathtt{i}\langle\cdot,\theta\rangle} S} \right) = \widehat{\varphi} \cdot \tau_\theta \widehat{S} = (2\pi)^n \widehat{\varphi} \sum_{\gamma\in\mathbb{Z}^n} \delta_{2\pi\gamma+\theta} \\ &= (2\pi)^n \sum_{\gamma\in\mathbb{Z}^n} \widehat{\varphi}(2\pi\gamma+\theta)\, \delta_{2\pi\gamma+\theta}. \end{aligned}$$

□

Above and in the rest of the paper for any $x \in \mathbb{R}^n$ and for any distribution u on $\mathbb{R}^n$, by $\tau_x u$ we shall denote the translation by x of u, i.e. $\tau_x u = u(\cdot - x) = \delta_x * u$.

As we already said the techniques of Coifman and Meyer, used in the study of Beurling algebras A_ω and B_ω (see [Co-Me] pp 7-10), can be adapted to the case of Sobolev spaces $\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$. An example is the following result.

Lemma 2.3. *Let $\mathbf{s}\in\mathbb{R}^j$. Let $\{u_\gamma\}_{\gamma\in\mathbb{Z}^n}$ be a a family of elements from $\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)\cap\mathcal{D}'_K(\mathbb{R}^n)$, where $K\subset\mathbb{R}^n$ is a compact subset such that $(K-K)\cap\mathbb{Z}^n=\{0\}$. Put*

$$u=\sum_{\gamma\in\mathbb{Z}^n}\tau_\gamma u_\gamma=\sum_{\gamma\in\mathbb{Z}^n}u_\gamma(\cdot-\gamma)=\sum_{\gamma\in\mathbb{Z}^n}\delta_\gamma * u_\gamma\in\mathcal{D}'(\mathbb{R}^n).$$

Then the following statements are equivalent:

(a) $u\in\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$.

(b) $\sum_{\gamma\in\mathbb{Z}^n}\|u_\gamma\|^2_{\mathcal{H}^{\mathbf{s}}}<\infty$.

Moreover, there is $C\geq 1$, which does not depend on the family $\{u_\gamma\}_{\gamma\in\mathbb{Z}^n}$, such that

$$C^{-1}\|u\|_{\mathcal{H}^{\mathbf{s}}}\leq\left(\sum_{\gamma\in\mathbb{Z}^n}\|u_\gamma\|^2_{\mathcal{H}^{\mathbf{s}}}\right)^{1/2}\leq C\|u\|_{\mathcal{H}^{\mathbf{s}}}. \tag{2.1}$$

Proof. Let us choose $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\varphi=1$ on K and $\mathtt{supp}\varphi=K'$ satisfies the condition $(K'-K')\cap\mathbb{Z}^n=\{0\}$. For $\theta\in[0,2\pi]^n$ we set

$$\begin{aligned}\varphi_\theta &= \sum_{\gamma\in\mathbb{Z}^n}\mathtt{e}^{\mathtt{i}\langle\gamma,\theta\rangle}\tau_\gamma\varphi=\sum_{\gamma\in\mathbb{Z}^n}\mathtt{e}^{\mathtt{i}\langle\gamma,\theta\rangle}\delta_\gamma*\varphi,\\ u_\theta &= \sum_{\gamma\in\mathbb{Z}^n}\mathtt{e}^{\mathtt{i}\langle\gamma,\theta\rangle}\tau_\gamma u_\gamma=\sum_{\gamma\in\mathbb{Z}^n}\mathtt{e}^{\mathtt{i}\langle\gamma,\theta\rangle}\delta_\gamma*u_\gamma.\end{aligned}$$

Since $(K'-K')\cap\mathbb{Z}^n=\{0\}$ we have

$$u_\theta=\varphi_\theta u,\qquad u=\varphi_\theta u_{-\theta}.$$

Step 1. Suppose first that the family $\{u_\gamma\}_{\gamma\in\mathbb{Z}^n}$ has only a finite number of non-zero terms and we shall prove in this case the estimate(2.1). Since u_θ, $u\in\mathcal{E}'\subset\mathcal{S}'$ it follows that

$$\widehat{u}_\theta=\nu_\theta*\widehat{u},\qquad\widehat{u}=\nu_\theta*\widehat{u}_{-\theta},$$

where $\nu_\theta=\widehat{\varphi}_\theta=(2\pi)^n\sum_{\gamma\in\mathbb{Z}^n}\widehat{\varphi}(2\pi\gamma+\theta)\delta_{2\pi\gamma+\theta}$ is a measure of rapid decay at ∞. Since $\widehat{u}_\theta$, $\widehat{u}\in\mathcal{C}^\infty_{pol}(\mathbb{R}^n)$ we get the pointwise equalities

$$\begin{aligned}\widehat{u}_\theta(\xi) &= (2\pi)^n\sum_{\gamma\in\mathbb{Z}^n}\widehat{\varphi}(2\pi\gamma+\theta)\widehat{u}(\xi-2\pi\gamma-\theta),\\ \widehat{u}(\xi) &= (2\pi)^n\sum_{\gamma\in\mathbb{Z}^n}\widehat{\varphi}(2\pi\gamma+\theta)\widehat{u}_{-\theta}(\xi-2\pi\gamma-\theta).\end{aligned}$$

By using Peetre's inequality we obtain

$$\begin{aligned}\langle\langle\xi\rangle\rangle^{\mathbf{s}}|\widehat{u}_\theta(\xi)|\leq 2^{|\mathbf{s}|_1/2}(2\pi)^n\sum_{\gamma\in\mathbb{Z}^n}&\langle\langle 2\pi\gamma+\theta\rangle\rangle^{|\mathbf{s}|}|\widehat{\varphi}(2\pi\gamma+\theta)|\\ &\cdot\langle\langle\xi-2\pi\gamma-\theta\rangle\rangle^{\mathbf{s}}|\widehat{u}(\xi-2\pi\gamma-\theta)|,\end{aligned}$$

and

$$\begin{aligned}\langle\langle\xi\rangle\rangle^{\mathbf{s}}\widehat{u}(\xi)\leq 2^{|\mathbf{s}|_1/2}(2\pi)^n\sum_{\gamma\in\mathbb{Z}^n}&\langle\langle 2\pi\gamma+\theta\rangle\rangle^{|\mathbf{s}|}|\widehat{\varphi}(2\pi\gamma+\theta)|\\ &\cdot\langle\langle\xi-2\pi\gamma-\theta\rangle\rangle^{\mathbf{s}}|\widehat{u}_{-\theta}(\xi-2\pi\gamma-\theta)|.\end{aligned}$$

From here we obtain further that

$$\begin{aligned}\|u_\theta\|_{\mathcal{H}^{\mathbf{s}}} &= \left\|\langle\langle\cdot\rangle\rangle^{\mathbf{s}}\,\widehat{u}_\theta\right\|_{L^2}\\ &\leq 2^{|\mathbf{s}|_1/2}(2\pi)^n\left(\sum_{\gamma\in\mathbb{Z}^n}\langle\langle 2\pi\gamma+\theta\rangle\rangle^{|\mathbf{s}|}\left|\widehat{\varphi}(2\pi\gamma+\theta)\right|\right)\left\|\langle\langle\cdot\rangle\rangle^{\mathbf{s}}\,\widehat{u}\right\|_{L^2}\\ &= 2^{|\mathbf{s}|_1/2}(2\pi)^n\left(\sum_{\gamma\in\mathbb{Z}^n}\langle\langle 2\pi\gamma+\theta\rangle\rangle^{|\mathbf{s}|}\left|\widehat{\varphi}(2\pi\gamma+\theta)\right|\right)\|u\|_{\mathcal{H}^{\mathbf{s}}}\\ &= C_{\mathbf{s},n,\varphi}\|u\|_{\mathcal{H}^{\mathbf{s}}}\end{aligned}$$

and

$$\begin{aligned}\|u\|_{\mathcal{H}^{\mathbf{s}}} &\leq 2^{|\mathbf{s}|_1/2}(2\pi)^n\left(\sum_{\gamma\in\mathbb{Z}^n}\langle\langle 2\pi\gamma+\theta\rangle\rangle^{|\mathbf{s}|}\left|\widehat{\varphi}(2\pi\gamma+\theta)\right|\right)\|u_{-\theta}\|_{\mathcal{H}^{\mathbf{s}}}\\ &= C_{\mathbf{s},n,\varphi}\|u_{-\theta}\|_{\mathcal{H}^{\mathbf{s}}}.\end{aligned}$$

The above estimates can be rewritten as

$$\begin{aligned}\int\langle\langle\xi\rangle\rangle^{2\mathbf{s}}\left|\widehat{u}_\theta(\xi)\right|^2\mathtt{d}\xi &\leq C^2_{\mathbf{s},n,\varphi}\|u\|^2_{\mathcal{H}^{\mathbf{s}}},\\ \|u\|^2_{\mathcal{H}^{\mathbf{s}}} &\leq C^2_{\mathbf{s},n,\varphi}\int\langle\langle\xi\rangle\rangle^{2\mathbf{s}}\left|\widehat{u}_{-\theta}(\xi)\right|^2\mathtt{d}\xi.\end{aligned}$$

On the other hand, the equality $u_\theta=\sum_{\gamma\in\mathbb{Z}^n}\mathtt{e}^{\mathtt{i}\langle\gamma,\theta\rangle}\tau_\gamma u_\gamma$ implies

$$\widehat{u}_\theta(\xi)=\sum_{\gamma\in\mathbb{Z}^n}\mathtt{e}^{\mathtt{i}\langle\gamma,\theta-\xi\rangle}\widehat{u}_\gamma(\xi)$$

with finite sum. The functions $\theta\to\widehat{u}_{\pm\theta}(\xi)$ are in $L^2([0,2\pi]^n)$ and

$$(2\pi)^{-n}\int_{[0,2\pi]^n}\left|\widehat{u}_{\pm\theta}(\xi)\right|^2\mathtt{d}\theta=\sum_{\gamma\in\mathbb{Z}^n}\left|\widehat{u}_\gamma(\xi)\right|^2.$$

Integrating with respect θ the above inequalities we get that

$$\begin{aligned}\sum_{\gamma\in\mathbb{Z}^n}\|u_\gamma\|^2_{\mathcal{H}^{\mathbf{s}}} &\leq C^2_{\mathbf{s},n,\varphi}\|u\|^2_{\mathcal{H}^{\mathbf{s}}},\\ \|u\|^2_{\mathcal{H}^{\mathbf{s}}} &\leq C^2_{\mathbf{s},n,\varphi}\sum_{\gamma\in\mathbb{Z}^n}\|u_\gamma\|^2_{\mathcal{H}^{\mathbf{s}}}.\end{aligned}$$

Step 2. The general case is obtained by approximation.

Suppose that $u\in\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$. Let $\psi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\psi=1$ on $B(0,1)$. Then $\psi^\varepsilon u\to u$ in $\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$ where $\psi^\varepsilon(x)=\psi(\varepsilon x)$, $0<\varepsilon\leq 1$, $x\in\mathbb{R}^n$. Also we have

$$\|\psi^\varepsilon u\|_{\mathcal{H}^{\mathbf{s}}}\leq C(s,n,\psi)\|u\|_{\mathcal{H}^{\mathbf{s}}},\quad 0<\varepsilon\leq 1,$$

where

$$\begin{aligned}C(s,n,\psi) &= (2\pi)^{-n}2^{|\mathbf{s}|_1/2}\sup_{0<\varepsilon\leq 1}\left(\int\langle\eta\rangle^{|\mathbf{s}|_1}\varepsilon^{-n}\left|\widehat{\psi}(\eta/\varepsilon)\right|\mathtt{d}\eta\right)\\ &= (2\pi)^{-n}2^{|\mathbf{s}|_1/2}\sup_{0<\varepsilon\leq 1}\left(\int\langle\varepsilon\eta\rangle^{|\mathbf{s}|_1}\left|\widehat{\psi}(\eta)\right|\mathtt{d}\eta\right)\\ &\leq (2\pi)^{-n}2^{|\mathbf{s}|_1/2}\left(\int\langle\eta\rangle^{|\mathbf{s}|_1}\left|\widehat{\psi}(\eta)\right|\mathtt{d}\eta\right).\end{aligned}$$

Let $m\in\mathbb{N}, m\geq 1$. Then there is ε_m such that for any $\varepsilon\in(0,\varepsilon_m]$ we have

$$\psi^{\varepsilon}u=\sum_{|\gamma|\leq m}\tau_{\gamma}u_{\gamma}+\sum_{finite}\tau_{\gamma}\left(\left(\tau_{-\gamma}\psi^{\varepsilon}\right)u_{\gamma}\right).$$

By the first part we get that

$$\sum_{|\gamma|\leq m}\left\|u_{\gamma}\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}\leq C_{\mathbf{s},n,\varphi}^{2}\left\|\psi^{\varepsilon}u\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}\leq C_{\mathbf{s},n,\varphi}^{2}C\left(s,n,\psi\right)^{2}\left\|u\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}.$$

Since m is arbitrary, it follows that $\sum_{\gamma\in\mathbb{Z}^n}\left\|u_{\gamma}\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}<\infty$. Further from

$$\sum_{|\gamma|\leq m}\left\|u_{\gamma}\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}\leq C_{\mathbf{s},n,\varphi}^{2}\left\|\psi^{\varepsilon}u\right\|_{\mathcal{H}^{\mathbf{s}}}^{2},\quad 0<\varepsilon\leq\varepsilon_m,$$

we obtain that

$$\sum_{|\gamma|\leq m}\left\|u_{\gamma}\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}\leq C_{\mathbf{s},n,\varphi}^{2}\left\|u\right\|_{\mathcal{H}^{\mathbf{s}}}^{2},\quad\forall m\in\mathbb{N}.$$

Hence

$$\sum_{\gamma\in\mathbb{Z}^n}\left\|u_{\gamma}\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}\leq C_{\mathbf{s},n,\varphi}^{2}\left\|u\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}.$$

Now suppose that $\sum_{\gamma\in\mathbb{Z}^n}\left\|u_{\gamma}\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}<\infty$. For $m\in\mathbb{N}$, $m\geq 1$ we put $u\left(m\right)=\sum_{|\gamma|\leq m}\tau_{\gamma}u_{\gamma}$. Then

$$\left\|u\left(m+p\right)-u\left(m\right)\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}\leq C_{\mathbf{s},n,\varphi}^{2}\sum_{m\leq|\gamma|\leq m+p}\left\|u_{\gamma}\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}$$

It follows that $\left\{u\left(m\right)\right\}_{m\geq 1}$ is a Cauchy sequence in $\mathcal{H}^{\mathbf{s}}\left(\mathbb{R}^n\right)$. Let $v\in\mathcal{H}^{\mathbf{s}}\left(\mathbb{R}^n\right)$ be such that $u\left(m\right)\to v$ in $\mathcal{H}^{\mathbf{s}}\left(\mathbb{R}^n\right)$. Since $u\left(m\right)\to u$ in $\mathcal{D}'\left(\mathbb{R}^n\right)$, it follows that $u=v$. Hence $u\left(m\right)\to u$ in $\mathcal{H}^{\mathbf{s}}\left(\mathbb{R}^n\right)$. Since we have

$$\left\|u\left(m\right)\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}\leq C_{\mathbf{s},n,\varphi}^{2}\sum_{|\gamma|\leq m}\left\|u_{\gamma}\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}\leq C_{\mathbf{s},n,\varphi}^{2}\sum_{\gamma\in\mathbb{Z}^n}\left\|u_{\gamma}\right\|_{\mathcal{H}^{\mathbf{s}}}^{2},\quad\forall m\in\mathbb{N}.$$

we obtain that

$$\left\|u\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}\leq C_{\mathbf{s},n,\varphi}^{2}\sum_{\gamma\in\mathbb{Z}^n}\left\|u_{\gamma}\right\|_{\mathcal{H}^{\mathbf{s}}}^{2}.$$

□

To use the previous result we need a convenient partition of unity. Let $N\in\mathbb{N}$ and $\{x_1,...,x_N\}\subset\mathbb{R}^n$ be such that

$$[0,1]^n\subset\left(x_1+\left[\frac{1}{3},\frac{2}{3}\right]^n\right)\cup...\cup\left(x_N+\left[\frac{1}{3},\frac{2}{3}\right]^n\right)$$

Let $\widetilde{h}\in\mathcal{C}_0^{\infty}\left(\mathbb{R}^n\right)$, $\widetilde{h}\geq 0$, be such that $\widetilde{h}=1$ on $\left[\frac{1}{3},\frac{2}{3}\right]^n$ and $\mathtt{supp}\widetilde{h}\subset\left[\frac{1}{4},\frac{3}{4}\right]^n$. Then

(a) $\widetilde{H}=\sum_{i=1}^{N}\sum_{\gamma\in\mathbb{Z}^n}\tau_{\gamma+x_i}\widetilde{h}\in\mathcal{BC}^{\infty}\left(\mathbb{R}^n\right)$ is $\mathbb{Z}^n$-periodic and $\widetilde{H}\geq 1$.

(b) $h_i=\frac{\tau_{x_i}\widetilde{h}}{\widetilde{H}}\in\mathcal{C}_0^{\infty}\left(\mathbb{R}^n\right)$, $h_i\geq 0$, $\mathtt{supp}h_i\subset x_i+\left[\frac{1}{4},\frac{3}{4}\right]^n=K_i$, $\left(K_i-K_i\right)\cap\mathbb{Z}^n=\{0\}$, $i=1,...,N$.

(c) $\chi_i=\sum_{\gamma\in\mathbb{Z}^n}\tau_{\gamma}h_i\in\mathcal{BC}^{\infty}\left(\mathbb{R}^n\right)$ is $\mathbb{Z}^n$-periodic, $i=1,...,N$ and $\sum_{i=1}^{N}\chi_i=1$.

(d) $h=\sum_{i=1}^{N}h_i\in\mathcal{C}_0^{\infty}\left(\mathbb{R}^n\right)$, $h\geq 0$, $\sum_{\gamma\in\mathbb{Z}^n}\tau_{\gamma}h=1$.

A first consequence of previous results is the next proposition.

Proposition 2.4. *Let* $\mathbf{s} \in \mathbb{R}^j$ *and* $m_{\mathbf{s}} = \left[|\mathbf{s}|_1 + \frac{n+1}{2}\right] + 1$. *Then*

$$\mathcal{BC}^{m_s}(\mathbb{R}^n) \cdot \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n).$$

Proof. Let $u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$. We use the partition of unity constructed above to obtain a decomposition of u satisfying the conditions of Lemma 2.3. Using Proposition 2.1 (c), it follows that $\chi_i u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$, $i = 1, ..., N$. We have

$$u = \sum_{i=1}^{N} \chi_i u$$

with $\chi_i u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$,

$$\chi_i u = \sum_{\gamma \in \mathbb{Z}^n} \tau_\gamma (h_i \tau_{-\gamma} u), \quad h_i \tau_{-\gamma} u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n) \cap \mathcal{D}'_{K_i}(\mathbb{R}^n),$$
$$(K_i - K_i) \cap \mathbb{Z}^n = \{0\}, \quad i = 1, ..., N.$$

So we can assume that $u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$ is of the form described in Lemma 2.3.

Let $\psi \in \mathcal{BC}^{m_s}(\mathbb{R}^n)$. Then

$$\psi u = \sum_{\gamma \in \mathbb{Z}^n} \psi \tau_\gamma u_\gamma = \sum_{\gamma \in \mathbb{Z}^n} \tau_\gamma \left(\psi_\gamma u_\gamma\right)$$

with $\psi_\gamma = \varphi(\tau_{-\gamma}\psi)$, where $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is the function considered in the proof of Lemma 2.3. We apply Lemma 2.3 and Proposition 2.1 (b) to obtain

$$\|\psi u\|_{\mathcal{H}^{\mathbf{s}}}^2 \leq C_{\mathbf{s},n,\varphi}^2 \sum_{\gamma \in \mathbb{Z}^n} \left\|\psi_\gamma u_\gamma\right\|_{\mathcal{H}^{\mathbf{s}}}^2$$

and

$$\begin{aligned} \left\|\psi_\gamma u_\gamma\right\|_{\mathcal{H}^{\mathbf{s}}} &\leq Cst \left(\sum_{|\alpha| \leq m_s} \|\partial^\alpha (\varphi(\tau_{-\gamma}\psi))\|_{L^2}\right) \|u_\gamma\|_{\mathcal{H}^{\mathbf{s}}} \\ &\leq Cst \, \|\varphi\|_{H^{m_s}} \|\psi\|_{\mathcal{BC}^{m_s}} \|u_\gamma\|_{\mathcal{H}^{\mathbf{s}}}, \quad \gamma \in \mathbb{Z}^n. \end{aligned}$$

Hence another application of Lemma 2.3 gives

$$\begin{aligned} \|\psi u\|_{\mathcal{H}^{\mathbf{s}}}^2 &\leq Cst \, \|\varphi\|_{H^{m_s}}^2 \|\psi\|_{\mathcal{BC}^{m_s}}^2 \sum_{\gamma \in \mathbb{Z}^n} \|u_\gamma\|_{\mathcal{H}^{\mathbf{s}}}^2 \\ &\leq Cst \, \|\varphi\|_{H^{m_s}}^2 \|\psi\|_{\mathcal{BC}^{m_s}}^2 \|u\|_{\mathcal{H}^{\mathbf{s}}}^2. \end{aligned}$$

□

Corollary 2.5. *Let* $\mathbf{s} \in \mathbb{R}^j$. *Then*

$$\mathcal{BC}^{\infty}(\mathbb{R}^n) \cdot \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n).$$

Lemma 2.6. *Let* λ_1, $\lambda_2 \geq 0$, $\lambda_1 + \lambda_2 > n/2$. *Then*

$$\langle\cdot\rangle_{\mathbb{R}^n}^{-2\lambda_1} * \langle\cdot\rangle_{\mathbb{R}^n}^{-2\lambda_2} \leq \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2(\lambda_1+\lambda_2)}\right\|_{L^1}$$

Proof. The case $\lambda_1 \cdot \lambda_2 = 0$ is trivial. Thus we may assume that λ_1, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 > n/2$. Then

$$\langle\cdot\rangle^{-2\lambda_j} \in L^{p_j}, \quad p_j = \frac{\lambda_1 + \lambda_2}{\lambda_j} > 1, \quad j = 1, 2.$$

Since $\frac{1}{p_1}+\frac{1}{p_2}=1$, by using Hölder's inequality we get

$$\langle\cdot\rangle_{\mathbb{R}^n}^{-2\lambda_1} * \langle\cdot\rangle_{\mathbb{R}^n}^{-2\lambda_2} \leq \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2\lambda_1}\right\|_{L^{p_1}} \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2\lambda_2}\right\|_{L^{p_2}}$$

with

$$\begin{aligned} \left\|\langle\cdot\rangle^{-2\lambda_j}\right\|_{L^{p_j}}^{p_j} &= \int \left[\left(1+|\xi|^2\right)^{-\lambda_j}\right]^{\frac{\lambda_1+\lambda_2}{\lambda_j}} \mathtt{d}\xi \\ &= \int \left(1+|\xi|^2\right)^{-\lambda_1-\lambda_2} \mathtt{d}\xi \\ &= \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2(\lambda_1+\lambda_2)}\right\|_{L^1}, \quad j=1,2. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle\cdot\rangle_{\mathbb{R}^n}^{-2\lambda_1} * \langle\cdot\rangle_{\mathbb{R}^n}^{-2\lambda_2} &\leq \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2\lambda_1}\right\|_{L^{p_1}} \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2\lambda_2}\right\|_{L^{p_2}} \\ &= \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2(\lambda_1+\lambda_2)}\right\|_{L^1}^{\frac{1}{p_1}+\frac{1}{p_2}} \\ &= \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2(\lambda_1+\lambda_2)}\right\|_{L^1}. \end{aligned}$$

□

Lemma 2.7. *Let $s,\ t\in\mathbb{R}$, $s+t>n/2$. For $\varepsilon\in(0,s+t-n/2)$ we put $\sigma(\varepsilon)=\min\{s,t,s+t-n/2-\varepsilon\}$. Then*

$$\langle\cdot\rangle_{\mathbb{R}^n}^{-2s} * \langle\cdot\rangle_{\mathbb{R}^n}^{-2t} \leq C(s,t,\varepsilon,n)\,\langle\cdot\rangle_{\mathbb{R}^n}^{-2\sigma(\varepsilon)}.$$

where

$$C(s,t,\varepsilon,n)=\begin{cases} 2^{2\sigma(\varepsilon)+1}\left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2(s+t-\sigma(\varepsilon))}\right\|_{L^1} & \text{if} \quad s,t\geq 0, \\ 2^{|\sigma(\varepsilon)|}\left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2(s+t)}\right\|_{L^1} & \text{if} \quad s<0 \text{ or } t<0. \end{cases}$$

Proof. Let us write σ for $\sigma(\varepsilon)$.

Step 1. The case $s,t\geq 0$. We have

$$\langle\cdot\rangle_{\mathbb{R}^n}^{-2s} * \langle\cdot\rangle_{\mathbb{R}^n}^{-2t}(\xi) = \int_{|\eta-\xi|\geq\frac{1}{2}|\xi|} \langle\xi-\eta\rangle_{\mathbb{R}^n}^{-2s}\langle\eta\rangle_{\mathbb{R}^n}^{-2t}\,\mathtt{d}\eta + \int_{|\eta-\xi|\leq\frac{1}{2}|\xi|} \langle\xi-\eta\rangle_{\mathbb{R}^n}^{-2s}\langle\eta\rangle_{\mathbb{R}^n}^{-2t}\,\mathtt{d}\eta$$

(a) If $|\eta-\xi|\geq\frac{1}{2}|\xi|$, then

$$\frac{1}{1+|\xi-\eta|^2} \leq \frac{4}{1+|\xi|^2} \Leftrightarrow \langle\xi-\eta\rangle_{\mathbb{R}^n}^{-1} \leq 2\langle\xi\rangle_{\mathbb{R}^n}^{-1}$$

and

$$\begin{aligned} \langle\xi-\eta\rangle_{\mathbb{R}^n}^{-2s} &= \langle\xi-\eta\rangle_{\mathbb{R}^n}^{-2\sigma}\cdot\langle\xi-\eta\rangle_{\mathbb{R}^n}^{-2(s-\sigma)} \\ &\leq 2^{2\sigma}\langle\xi\rangle_{\mathbb{R}^n}^{-2\sigma}\langle\xi-\eta\rangle_{\mathbb{R}^n}^{-2(s-\sigma)} \end{aligned}$$

Since $s-\sigma+t = s+t-n/2-\varepsilon-\sigma+n/2+\varepsilon \geq n/2+\varepsilon > n/2$, the previous lemma allows to evaluate the integral on the domain $|\eta-\xi| \geq \frac{1}{2}|\xi|$

$$\begin{aligned}\int_{|\eta-\xi|\geq\frac{1}{2}|\xi|} \langle\xi-\eta\rangle_{\mathbb{R}^n}^{-2s} \langle\eta\rangle_{\mathbb{R}^n}^{-2t} \mathtt{d}\eta &\leq 2^{2\sigma} \langle\xi\rangle_{\mathbb{R}^n}^{-2\sigma} \int_{|\eta-\xi|\geq\frac{1}{2}|\xi|} \langle\xi-\eta\rangle_{\mathbb{R}^n}^{-2(s-\sigma)} \langle\eta\rangle_{\mathbb{R}^n}^{-2t} \mathtt{d}\eta \\ &\leq 2^{2\sigma} \langle\xi\rangle_{\mathbb{R}^n}^{-2\sigma} \left(\langle\cdot\rangle_{\mathbb{R}^n}^{-2(s-\sigma)} * \langle\cdot\rangle_{\mathbb{R}^n}^{-2t}\right)(\xi) \\ &\leq 2^{2\sigma} \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2(s+t-\sigma)}\right\|_{L^1} \langle\xi\rangle_{\mathbb{R}^n}^{-2\sigma}\end{aligned}$$

(b) If $|\eta-\xi| \leq \frac{1}{2}|\xi|$, then $|\eta| \geq |\xi| - |\eta-\xi| \geq \frac{1}{2}|\xi|$. We can therefore use (a) to evaluate the integral on the domain $|\eta-\xi| \leq \frac{1}{2}|\xi|$. It follows that

$$\begin{aligned}\int_{|\eta-\xi|\leq\frac{1}{2}|\xi|} \langle\xi-\eta\rangle_{\mathbb{R}^n}^{-2s} \langle\eta\rangle_{\mathbb{R}^n}^{-2t} \mathtt{d}\eta &\leq \int_{|\eta|\geq\frac{1}{2}|\xi|} \langle\xi-\eta\rangle_{\mathbb{R}^n}^{-2s} \langle\eta\rangle_{\mathbb{R}^n}^{-2t} \mathtt{d}\eta \\ &= \int_{|\zeta-\xi|\geq\frac{1}{2}|\xi|} \langle\zeta\rangle_{\mathbb{R}^n}^{-2s} \langle\xi-\zeta\rangle_{\mathbb{R}^n}^{-2t} \mathtt{d}\zeta \\ &\leq 2^{2\sigma} \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2(s+t-\sigma)}\right\|_{L^1} \langle\xi\rangle_{\mathbb{R}^n}^{-2\sigma}\end{aligned}$$

(c) From (a) and (b) we obtain

$$\langle\cdot\rangle_{\mathbb{R}^n}^{-2s} * \langle\cdot\rangle_{\mathbb{R}^n}^{-2t} \leq 2^{2\sigma+1} \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2(s+t-\sigma)}\right\|_{L^1} \langle\cdot\rangle_{\mathbb{R}^n}^{-2\sigma}.$$

Step 2. Next we consider the case $s<0$ or $t<0$. If $s<0$ and $s+t>n/2$, then $\sigma = s$. In this case we use Peetre's inequality to obtain:

$$\begin{aligned}\langle\cdot\rangle_{\mathbb{R}^n}^{-2s} * \langle\cdot\rangle_{\mathbb{R}^n}^{-2t}(\xi) &= \int \langle\xi-\eta\rangle_{\mathbb{R}^n}^{-2s} \langle\eta\rangle_{\mathbb{R}^n}^{-2t} \mathtt{d}\eta \\ &\leq 2^{|s|} \int \langle\xi\rangle_{\mathbb{R}^n}^{-2s} \langle\eta\rangle_{\mathbb{R}^n}^{-2(s+t)} \mathtt{d}\eta \\ &= 2^{|\sigma|} \left\|\langle\cdot\rangle_{\mathbb{R}^n}^{-2(s+t)}\right\|_{L^1} \langle\xi\rangle_{\mathbb{R}^n}^{-2\sigma}\end{aligned}$$

The case $t<0$ can be treated similarly. □

Since $\langle\langle\cdot\rangle\rangle^{\mathbf{s}} = \langle\cdot\rangle_{\mathbb{R}^{n_1}}^{s_1} \otimes ... \otimes \langle\cdot\rangle_{\mathbb{R}^{n_j}}^{s_j}$, $\mathbf{s} = (s_1, ..., s_j) \in \mathbb{R}^j$ we obtain

Corollary 2.8. *Let* $\mathbf{s}$, $\mathbf{t}$, $\boldsymbol{\varepsilon}$, $\sigma(\boldsymbol{\varepsilon}) \in \mathbb{R}^j$ *such that,* $s_l + t_l > n_l/2$, $0 < \varepsilon_l < s_l + t_l - n_l/2$, $\sigma_l(\boldsymbol{\varepsilon}) = \sigma_l(\varepsilon_l) = \min\{s_l, t_l, s_l+t_l-n_l/2-\varepsilon_l\}$ *for any* $l \in \{1, ..., j\}$. *Then there is* $C(\mathbf{s}, \mathbf{t}, \boldsymbol{\varepsilon}, n) > 0$ *such that*

$$\langle\langle\cdot\rangle\rangle^{-2\mathbf{s}} * \langle\langle\cdot\rangle\rangle^{-2\mathbf{t}} \leq C(\mathbf{s}, \mathbf{t}, \boldsymbol{\varepsilon}, n) \langle\langle\cdot\rangle\rangle^{-2\boldsymbol{\sigma}(\boldsymbol{\varepsilon})}.$$

Proposition 2.9. *Let* $\mathbf{s}$, $\mathbf{t}$, $\boldsymbol{\varepsilon}$, $\sigma(\boldsymbol{\varepsilon}) \in \mathbb{R}^j$ *such that,* $s_l + t_l > n_l/2$, $0 < \varepsilon_l < s_l + t_l - n_l/2$, $\sigma_l(\boldsymbol{\varepsilon}) = \sigma_l(\varepsilon_l) = \min\{s_l, t_l, s_l+t_l-n_l/2-\varepsilon_l\}$ *for any* $l \in \{1, ..., j\}$. *Then*

$$\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n) \cdot \mathcal{H}^{\mathbf{t}}(\mathbb{R}^n) \subset \mathcal{H}^{\boldsymbol{\sigma}(\boldsymbol{\varepsilon})}(\mathbb{R}^n)$$

Proof. Let us write σ for $\sigma(\varepsilon)$. Let $u, v \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}\|u\cdot v\|_{\mathcal{H}^{\boldsymbol{\sigma}}}^2 = \left\|\langle\langle\cdot\rangle\rangle^{\boldsymbol{\sigma}} \widehat{u\cdot v}\right\|_{L^2}^2 = \int \left|\langle\langle\xi\rangle\rangle^{\boldsymbol{\sigma}} \widehat{u\cdot v}(\xi)\right|^2 \mathtt{d}\xi \\ = (2\pi)^{-2n} \int \left|\langle\langle\xi\rangle\rangle^{\boldsymbol{\sigma}} \widehat{u} * \widehat{v}(\xi)\right|^2 \mathtt{d}\xi\end{aligned}$$

By using Schwarz's inequality and the above corollary we can estimate the integrand as follows

$$\begin{aligned}\left|\langle\langle\xi\rangle\rangle^{\boldsymbol{\sigma}}\,\widehat{u}*\widehat{v}\,(\xi)\right|^2 &\leq \left(\int\left|\langle\langle\eta\rangle\rangle^{\mathbf{s}}\,\widehat{u}\,(\eta)\right|\left|\langle\langle\xi-\eta\rangle\rangle^{\mathbf{t}}\,\widehat{v}\,(\xi-\eta)\right|\frac{\langle\langle\xi\rangle\rangle^{\boldsymbol{\sigma}}}{\langle\langle\eta\rangle\rangle^{\mathbf{s}}\langle\langle\xi-\eta\rangle\rangle^{\mathbf{t}}}\mathtt{d}\eta\right)^2\\ &\leq \left(\int\left|\langle\langle\eta\rangle\rangle^{\mathbf{s}}\,\widehat{u}\,(\eta)\right|^2\left|\langle\langle\xi-\eta\rangle\rangle^{\mathbf{t}}\,\widehat{v}\,(\xi-\eta)\right|^2\mathtt{d}\eta\right)\left(\int\frac{\langle\langle\xi\rangle\rangle^{2\boldsymbol{\sigma}}}{\langle\langle\eta\rangle\rangle^{2\mathbf{s}}\langle\langle\xi-\eta\rangle\rangle^{2\mathbf{t}}}\mathtt{d}\eta\right)\\ &\leq C\left(\mathbf{s},\mathbf{t},\boldsymbol{\varepsilon},n\right)\int\left|\langle\langle\eta\rangle\rangle^{\mathbf{s}}\,\widehat{u}\,(\eta)\right|^2\left|\langle\langle\xi-\eta\rangle\rangle^{\mathbf{t}}\,\widehat{v}\,(\xi-\eta)\right|^2\mathtt{d}\eta\end{aligned}$$

Hence

$$\begin{aligned}\|u\cdot v\|^2_{\mathcal{H}^{\boldsymbol{\sigma}}} &\leq C'\left(\mathbf{s},\mathbf{t},\boldsymbol{\varepsilon},n\right)\int\left(\int\left|\langle\langle\eta\rangle\rangle^{\mathbf{s}}\,\widehat{u}\,(\eta)\right|^2\left|\langle\langle\xi-\eta\rangle\rangle^{\mathbf{t}}\,\widehat{v}\,(\xi-\eta)\right|^2\mathtt{d}\eta\right)\mathtt{d}\xi\\ &= C'\left(\mathbf{s},\mathbf{t},\boldsymbol{\varepsilon},n\right)\|u\|^2_{\mathcal{H}^{\mathbf{s}}}\|v\|^2_{\mathcal{H}^{\mathbf{t}}}\end{aligned}$$

To conclude we use the fact that $\mathcal{S}\left(\mathbb{R}^n\right)$ is dense in any $\mathcal{H}^{\mathbf{m}}\left(\mathbb{R}^n\right)$. □

Corollary 2.10. *Let* $\mathbf{s}\in\mathbb{R}^j$. *If* $s_1>n_1/2,...,s_j>n_j/2$, *then* $\mathcal{H}^{\mathbf{s}}\left(\mathbb{R}^n\right)$ *is a Banach algebra.*

3. Kato-Sobolev spaces $\mathcal{K}^{\mathbf{s}}_p\left(\mathbb{R}^n\right)$

We begin by proving some results that will be useful later. Let $\varphi,\psi\in\mathcal{C}_0^\infty\left(\mathbb{R}^n\right)$ (or $\varphi,\psi\in\mathcal{S}\left(\mathbb{R}^n\right)$). Then the maps

$$\begin{aligned}\mathbb{R}^n\times\mathbb{R}^n &\ni (x,y)\xrightarrow{f}\varphi(x)\,\psi(x-y)=\left(\varphi\tau_y\psi\right)(x)\in\mathbb{C},\\ \mathbb{R}^n\times\mathbb{R}^n &\ni (x,y)\xrightarrow{g}\varphi(y)\,\psi(x-y)=\varphi(y)\left(\tau_y\psi\right)(x)\in\mathbb{C},\end{aligned}$$

are in $\mathcal{C}_0^\infty\left(\mathbb{R}^n\times\mathbb{R}^n\right)$ (respectively in $\mathcal{S}\left(\mathbb{R}^n\times\mathbb{R}^n\right)$). To see this we note that

$$f=(\varphi\otimes\psi)\circ T,\quad g=(\varphi\otimes\psi)\circ S$$

where

$$\begin{aligned}T &: \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^n,\quad T(x,y)=(x,x-y),\quad T\equiv\begin{pmatrix}\mathtt{I} & 0\\ \mathtt{I} & -\mathtt{I}\end{pmatrix},\\ S &: \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^n,\quad S(x,y)=(y,x-y),\quad S\equiv\begin{pmatrix}0 & \mathtt{I}\\ \mathtt{I} & -\mathtt{I}\end{pmatrix}.\end{aligned}$$

Let $u\in\mathcal{D}'\left(\mathbb{R}^n\right)$ (or $u\in\mathcal{S}'\left(\mathbb{R}^n\right)$). Then using Fubini theorem for distributions we get

$$\begin{aligned}\langle u\otimes 1,f\rangle &= \langle (u\otimes 1)(x,y),\varphi(x)\,\psi(x-y)\rangle\\ &= \langle u(x),\langle 1(y),\varphi(x)\,\psi(x-y)\rangle\rangle\\ &= \langle u(x),\varphi(x)\,\langle 1(y),\psi(x-y)\rangle\rangle\\ &= \left\langle u(x),\varphi(x)\int\psi(x-y)\,\mathtt{d}y\right\rangle\\ &= \left(\int\psi\right)\langle u,\varphi\rangle\end{aligned}$$

and

$$\begin{aligned}\langle u\otimes 1,f\rangle &= \langle 1\left(y\right),\langle u\left(x\right),\varphi\left(x\right)\psi\left(x-y\right)\rangle\rangle\\ &= \int\langle u,\varphi\tau_y\psi\rangle\,\mathtt{d}y.\end{aligned}$$

It follows that

$$\left(\int\psi\right)\langle u,\varphi\rangle=\int\langle u,\varphi\tau_y\psi\rangle\,\mathtt{d}y$$

valid for

(i) $u\in\mathcal{D}'\left(\mathbb{R}^n\right)$, $\varphi,\psi\in\mathcal{C}_0^\infty\left(\mathbb{R}^n\right)$;
(ii) $u\in\mathcal{S}'\left(\mathbb{R}^n\right)$, $\varphi,\psi\in\mathcal{S}\left(\mathbb{R}^n\right)$.

We also have

$$\begin{aligned}\langle u\otimes 1,g\rangle &= \langle\left(u\otimes 1\right)\left(x,y\right),\varphi\left(y\right)\psi\left(x-y\right)\rangle\\ &= \langle u\left(x\right),\langle 1\left(y\right),\varphi\left(y\right)\psi\left(x-y\right)\rangle\rangle\\ &= \langle u\left(x\right),\left(\varphi*\psi\right)\left(x\right)\rangle\\ &= \langle u,\varphi*\psi\rangle\end{aligned}$$

and

$$\begin{aligned}\langle u\otimes 1,g\rangle &= \langle 1\left(y\right),\langle u\left(x\right),\varphi\left(y\right)\psi\left(x-y\right)\rangle\rangle\\ &= \int\varphi\left(y\right)\langle u,\tau_y\psi\rangle\,\mathtt{d}y.\end{aligned}$$

Hence

$$\langle u,\varphi*\psi\rangle=\int\varphi\left(y\right)\langle u,\tau_y\psi\rangle\,\mathtt{d}y$$

true for

(i) $u\in\mathcal{D}'\left(\mathbb{R}^n\right)$, $\varphi,\psi\in\mathcal{C}_0^\infty\left(\mathbb{R}^n\right)$;
(ii) $u\in\mathcal{S}'\left(\mathbb{R}^n\right)$, $\varphi,\psi\in\mathcal{S}\left(\mathbb{R}^n\right)$.

Lemma 3.1. *Let $\varphi,\psi\in\mathcal{C}_0^\infty\left(\mathbb{R}^n\right)$ (or $\varphi,\psi\in\mathcal{S}\left(\mathbb{R}^n\right)$) and $u\in\mathcal{D}'\left(\mathbb{R}^n\right)$ (or $u\in\mathcal{S}'\left(\mathbb{R}^n\right)$). Then*

$$\left(\int\psi\right)\langle u,\varphi\rangle=\int\langle u,\varphi\tau_y\psi\rangle\,\mathtt{d}y \tag{3.1}$$

$$\langle u,\varphi*\psi\rangle=\int\varphi\left(y\right)\langle u,\tau_y\psi\rangle\,\mathtt{d}y \tag{3.2}$$

If $\varepsilon_1,...,\varepsilon_n$ is a basis in $\mathbb{R}^n$, we say that $\Gamma=\oplus_{j=1}^n\mathbb{Z}\varepsilon_j$ is a lattice.

Let $\Gamma\subset\mathbb{R}^n$ be a lattice. Let $\psi\in\mathcal{S}\left(\mathbb{R}^n\right)$. Then $\sum_{\gamma\in\Gamma}\tau_\gamma\psi=\sum_{\gamma\in\Gamma}\psi\left(\cdot-\gamma\right)$ is uniformly convergent on compact subsets of $\mathbb{R}^n$. Since $\partial^\alpha\psi\in\mathcal{S}\left(\mathbb{R}^n\right)$, it follows that there is $\Psi\in\mathcal{C}^\infty\left(\mathbb{R}^n\right)$ such that

$$\Psi=\sum_{\gamma\in\Gamma}\tau_\gamma\psi=\sum_{\gamma\in\Gamma}\psi\left(\cdot-\gamma\right)\quad\textit{in }\mathcal{C}^\infty\left(\mathbb{R}^n\right).$$

Moreover we have $\tau_\gamma\Psi=\Psi\left(\cdot-\gamma\right)=\Psi$ for any $\gamma\in\Gamma$. From here we obtain that $\Psi\in\mathcal{BC}^\infty\left(\mathbb{R}^n\right)$. If $\Psi\left(y\right)\neq 0$ for any $y\in\mathbb{R}^n$, then $\frac{1}{\Psi}\in\mathcal{BC}^\infty\left(\mathbb{R}^n\right)$.

Let $\varphi\in\mathcal{S}\left(\mathbb{R}^n\right)$. Then

$$\varphi\Psi=\sum_{\gamma\in\Gamma}\varphi\left(\tau_\gamma\psi\right)$$

with the series convergent in $\mathcal{S}(\mathbb{R}^n)$. Indeed we have

$$\begin{aligned}
&\sum_{\gamma\in\Gamma} \langle x\rangle^k \left|\partial^\alpha \varphi(x)\, \partial^\beta \psi(x-\gamma)\right| \\
&\qquad\leq \sup_y \langle y\rangle^{n+1} \left|\partial^\beta \psi(y)\right| \sum_{\gamma\in\Gamma} \langle x\rangle^k \left|\partial^\alpha \varphi(x)\, \langle x-\gamma\rangle^{-n-1}\right| \\
&\qquad\leq 2^{\frac{n+1}{2}} \sup_y \langle y\rangle^{n+1} \left|\partial^\beta \psi(y)\right| \sup_z \langle z\rangle^{k+n+1} \left|\partial^\alpha \varphi(z)\right| \sum_{\gamma\in\Gamma} \langle \gamma\rangle^{-n-1}.
\end{aligned}$$

This estimate proves the convergence of the series in $\mathcal{S}(\mathbb{R}^n)$. Let χ be the sum of the series $\sum_{\gamma\in\Gamma} \varphi(\tau_\gamma\psi)$ in $\mathcal{S}(\mathbb{R}^n)$. Then for any $y\in\mathbb{R}^n$ we have

$$\begin{aligned}
\chi(y) &= \langle \delta_y, \chi\rangle = \left\langle \delta_y, \sum_{\gamma\in\Gamma} \varphi(\tau_\gamma\psi)\right\rangle \\
&= \sum_{\gamma\in\Gamma} \langle \delta_y, \varphi(\tau_\gamma\psi)\rangle = \sum_{\gamma\in\Gamma} \varphi(y)\, \psi(y-\gamma) \\
&= \varphi(y)\, \Psi(y).
\end{aligned}$$

So $\varphi\Psi = \sum_{\gamma\in\Gamma} \varphi(\tau_\gamma\psi)$ in $\mathcal{S}(\mathbb{R}^n)$.

If $\psi,\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is replaced by $\mathcal{C}_0^\infty(\mathbb{R}^n)$, then the previous observations are trivial.

Lemma 3.2. *Let $u\in\mathcal{D}'(\mathbb{R}^n)$ (or $u\in\mathcal{S}'(\mathbb{R}^n)$) and $\psi,\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\psi,\varphi\in\mathcal{S}(\mathbb{R}^n)$). Then $\Psi=\sum_{\gamma\in\Gamma}\tau_\gamma\psi\in\mathcal{BC}^\infty(\mathbb{R}^n)$ is Γ-periodic and*

$$\langle u, \Psi\varphi\rangle = \sum_{\gamma\in\Gamma} \langle u, (\tau_\gamma\psi)\,\varphi\rangle. \tag{3.3}$$

Lemma 3.3. (a) *Let $\chi\in\mathcal{S}(\mathbb{R}^n)$ and $u\in\mathcal{S}'(\mathbb{R}^n)$. Then $\widehat{\chi u}\in\mathcal{S}'(\mathbb{R}^n)\cap\mathcal{C}_{pol}^\infty(\mathbb{R}^n)$. In fact we have*

$$\widehat{\chi u}(\xi) = \left\langle \mathtt{e}^{-\mathtt{i}\langle\cdot,\xi\rangle} u, \chi\right\rangle = \left\langle u, \mathtt{e}^{-\mathtt{i}\langle\cdot,\xi\rangle}\chi\right\rangle, \quad \xi\in\mathbb{R}^n.$$

(b) *Let $u\in\mathcal{D}'(\mathbb{R}^n)$ (or $u\in\mathcal{S}'(\mathbb{R}^n)$) and $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\chi\in\mathcal{S}(\mathbb{R}^n)$). Then*

$$\mathbb{R}^n\times\mathbb{R}^n \ni (y,\xi) \to \widehat{u\tau_y\chi}(\xi) = \left\langle u, \mathtt{e}^{-\mathtt{i}\langle\cdot,\xi\rangle}\chi(\cdot - y)\right\rangle \in\mathbb{C}$$

is a $\mathcal{C}^\infty$-function.

Proof. Let $q:\mathbb{R}^n_x\times\mathbb{R}^n_\xi\to\mathbb{R}$, $q(x,\xi)=\langle x,\xi\rangle$. Then $\mathtt{e}^{-\mathtt{i}q}(u\otimes 1)\in\mathcal{S}'\left(\mathbb{R}^n_x\times\mathbb{R}^n_\xi\right)$. If $\varphi\in\mathcal{S}\left(\mathbb{R}^n_\xi\right)$, then we have

$$\begin{aligned}
\left\langle \mathtt{e}^{-\mathtt{i}q}(u\otimes 1), \chi\otimes\varphi\right\rangle &= \left\langle u\otimes 1, \mathtt{e}^{-\mathtt{i}q}(\chi\otimes\varphi)\right\rangle \\
&= \left\langle u(x), \left\langle 1(\xi), \mathtt{e}^{-\mathtt{i}q(x,\xi)}\chi(x)\,\varphi(\xi)\right\rangle\right\rangle \\
&= \left\langle u(x), \chi(x)\left\langle 1(\xi), \mathtt{e}^{-\mathtt{i}\langle x,\xi\rangle}\varphi(\xi)\right\rangle\right\rangle \\
&= \langle u, \chi\widehat{\varphi}\rangle = \langle \widehat{\chi u}, \varphi\rangle
\end{aligned}$$

and

$$\begin{aligned}\langle \widehat{\chi u}, \varphi \rangle &= \left\langle \mathtt{e}^{-\mathtt{i}q} \left(u \otimes 1\right), \chi \otimes \varphi \right\rangle \\ &= \left\langle 1\left(\xi\right), \left\langle u\left(x\right), \mathtt{e}^{-\mathtt{i}\langle x,\xi\rangle} \chi\left(x\right) \varphi\left(\xi\right) \right\rangle \right\rangle \\ &= \left\langle 1\left(\xi\right), \varphi\left(\xi\right) \left\langle u, \mathtt{e}^{-\mathtt{i}\langle \cdot,\xi\rangle} \chi \right\rangle \right\rangle \\ &= \left\langle 1\left(\xi\right), \varphi\left(\xi\right) \left\langle \mathtt{e}^{-\mathtt{i}\langle \cdot,\xi\rangle} u, \chi \right\rangle \right\rangle \\ &= \int \varphi\left(\xi\right) \left\langle \mathtt{e}^{-\mathtt{i}\langle \cdot,\xi\rangle} u, \chi \right\rangle \mathtt{d}\xi\end{aligned}$$

This proves that

$$\widehat{\chi u}\left(\xi\right) = \left\langle \mathtt{e}^{-\mathtt{i}\langle \cdot,\xi\rangle} u, \chi \right\rangle, \quad \xi \in \mathbb{R}^n.$$

□

Let $u \in \mathcal{D}'\left(\mathbb{R}^n\right)$ (or $u \in \mathcal{S}'\left(\mathbb{R}^n\right)$) and $\chi \in \mathcal{C}_0^\infty\left(\mathbb{R}^n\right) \smallsetminus 0$ (or $\chi \in \mathcal{S}\left(\mathbb{R}^n\right) \smallsetminus 0$). Let $\widetilde{\chi} \in \mathcal{C}_0^\infty\left(\mathbb{R}^n\right)$ (or $\widetilde{\chi} \in \mathcal{S}\left(\mathbb{R}^n\right)$)and $\varphi \in \mathcal{C}_0^\infty\left(\mathbb{R}^n\right)$. By using (3.1) we get

$$\begin{aligned}\langle u\tau_z\widetilde{\chi}, \varphi\rangle &= \frac{1}{\left\|\chi\right\|_{L^2}^2} \int \langle u\tau_z\widetilde{\chi}, \left(\tau_y\chi\right)\left(\tau_y\overline{\chi}\right)\varphi\rangle \,\mathtt{d}y \\ &= \frac{1}{\left\|\chi\right\|_{L^2}^2} \int \langle u\tau_y\chi, \left(\tau_z\widetilde{\chi}\right)\left(\tau_y\overline{\chi}\right)\varphi\rangle \,\mathtt{d}y,\end{aligned}$$

$$\left|\langle u\tau_z\widetilde{\chi}, \varphi\rangle\right| \le \frac{1}{\left\|\chi\right\|_{L^2}^2} \int \left\|u\tau_y\chi\right\|_{\mathcal{H}^{\mathtt{s}}} \left\|\left(\tau_z\widetilde{\chi}\right)\left(\tau_y\overline{\chi}\right)\varphi\right\|_{\mathcal{H}^{-\mathtt{s}}} \mathtt{d}y.$$

Let $\Gamma \subset \mathbb{R}^n$ be a lattice. Let $u \in \mathcal{D}'\left(\mathbb{R}^n\right)$ (or $u \in \mathcal{S}'\left(\mathbb{R}^n\right)$) and let $\chi \in \mathcal{C}_0^\infty\left(\mathbb{R}^n\right)$ (or $\chi \in \mathcal{S}\left(\mathbb{R}^n\right)$) be such that

$$\Psi = \Psi_{\Gamma,\chi} = \sum_{\gamma\in\Gamma} \left|\tau_\gamma\chi\right|^2 > 0.$$

Then $\Psi, \frac{1}{\Psi} \in \mathcal{BC}^\infty\left(\mathbb{R}^n\right)$ and both are Γ-periodic. Let $\widetilde{\chi} \in \mathcal{C}_0^\infty\left(\mathbb{R}^n\right)$ (or $\widetilde{\chi} \in \mathcal{S}\left(\mathbb{R}^n\right)$). Using (3.3) we obtain that

$$\begin{aligned}\langle u\tau_z\widetilde{\chi}, \varphi\rangle &= \sum_{\gamma\in\Gamma} \left\langle u\tau_\gamma\chi, \frac{1}{\Psi}\left(\tau_\gamma\overline{\chi}\right)\left(\tau_z\widetilde{\chi}\right)\varphi \right\rangle, \\ \left|\langle u\tau_z\widetilde{\chi}, \varphi\rangle\right| &\le \sum_{\gamma\in\Gamma} \left\|u\tau_\gamma\chi\right\|_{\mathcal{H}^{\mathtt{s}}} \left\|\frac{1}{\Psi}\left(\tau_\gamma\overline{\chi}\right)\left(\tau_z\widetilde{\chi}\right)\varphi\right\|_{\mathcal{H}^{-\mathtt{s}}} \\ &\le C_\Psi \sum_{\gamma\in\Gamma} \left\|u\tau_\gamma\chi\right\|_{\mathcal{H}^{\mathtt{s}}} \left\|\left(\tau_\gamma\overline{\chi}\right)\left(\tau_z\widetilde{\chi}\right)\varphi\right\|_{\mathcal{H}^{-\mathtt{s}}}.\end{aligned}$$

In the last inequality we used the Proposition 2.4 and the fact that $\frac{1}{\Psi} \in \mathcal{BC}^\infty\left(\mathbb{R}^n\right)$.

If (Y,μ) is either $\mathbb{R}^n$ with Lebesgue measure or Γ with the counting measure, then the previous estimates can be written as:

$$\left|\langle u\tau_z\widetilde{\chi}, \varphi\rangle\right| \le Cst \int_Y \left\|u\tau_y\chi\right\|_{\mathcal{H}^{\mathtt{s}}} \left\|\left(\tau_z\widetilde{\chi}\right)\left(\tau_y\overline{\chi}\right)\varphi\right\|_{\mathcal{H}^{-\mathtt{s}}} \mathtt{d}\mu\left(y\right)$$

We shall use Proposition 2.4 to estimate $\|(\tau_z\widetilde{\chi})(\tau_y\overline{\chi})\varphi\|_{\mathcal{H}^{-\mathbf{s}}}$. Let us write $m_{\mathbf{s}}$ for $\left[|\mathbf{s}|_1+\frac{n+1}{2}\right]+1$. Then we have

$$\|(\tau_z\widetilde{\chi})(\tau_y\overline{\chi})\varphi\|_{\mathcal{H}^{-\mathbf{s}}}\leq Cst\sup_{|\alpha+\beta|\leq m_{\mathbf{s}}}\left|\left((\tau_z\partial^\alpha\widetilde{\chi})\left(\tau_y\partial^\beta\overline{\chi}\right)\right)\right|\|\varphi\|_{\mathcal{H}^{-\mathbf{s}}}.$$

For any $N\in\mathbb{N}$ there is a continuous seminorm $p=p_{N,\mathbf{s}}$ on $\mathcal{S}(\mathbb{R}^n)$ so that

$$\begin{aligned}\left|(\tau_z\partial^\alpha\widetilde{\chi})\left(\tau_y\partial^\beta\overline{\chi}\right)(x)\right| &\leq p(\widetilde{\chi})\,p(\chi)\langle x-z\rangle^{-2N}\langle x-y\rangle^{-2N}\\ &\leq 2^Np(\widetilde{\chi})\,p(\chi)\langle 2x-z-y\rangle^{-N}\langle z-y\rangle^{-N}\\ &\leq 2^Np(\widetilde{\chi})\,p(\chi)\langle z-y\rangle^{-N},\quad |\alpha+\beta|\leq m_{\mathbf{s}}.\end{aligned}$$

Here we used the inequality

$$\langle X\rangle^{-2N}\langle Y\rangle^{-2N}\leq 2^N\langle X+Y\rangle^{-N}\langle X-Y\rangle^{-N},\quad X,Y\in\mathbb{R}^m$$

which is a consequence of Peetre's inequality:

$$\left.\begin{aligned}\langle X+Y\rangle^N\leq 2^{\frac{N}{2}}\langle X\rangle^N\langle Y\rangle^N\\ \langle X-Y\rangle^N\leq 2^{\frac{N}{2}}\langle X\rangle^N\langle Y\rangle^N\end{aligned}\right\}\Rightarrow\langle X+Y\rangle^N\langle X-Y\rangle^N\leq 2^N\langle X\rangle^{2N}\langle Y\rangle^{2N}$$

Hence

$$\sup_{|\alpha+\beta|\leq m_{\mathbf{s}}}\left|\left((\tau_z\partial^\alpha\widetilde{\chi})\left(\tau_y\partial^\beta\overline{\chi}\right)\right)\right|\leq 2^Np_{N,\mathbf{s}}(\widetilde{\chi})\,p_{N,\mathbf{s}}(\chi)\langle z-y\rangle^{-N},$$

$$\|(\tau_z\widetilde{\chi})(\tau_y\overline{\chi})\varphi\|_{\mathcal{H}^{-\mathbf{s}}}\leq C(N,\mathbf{s},\chi,\widetilde{\chi})\langle z-y\rangle^{-N}\|\varphi\|_{\mathcal{H}^{-\mathbf{s}}},$$

$$|\langle u\tau_z\widetilde{\chi},\varphi\rangle|\leq C(N,\mathbf{s},\chi,\widetilde{\chi})\left(\int_Y\|u\tau_y\chi\|_{\mathcal{H}^{\mathbf{s}}}\langle z-y\rangle^{-N}\mathtt{d}\mu(y)\right)\|\varphi\|_{\mathcal{H}^{-\mathbf{s}}}.$$

The last estimate implies that

$$\|u\tau_z\widetilde{\chi}\|_{\mathcal{H}^{\mathbf{s}}}\leq C(N,\mathbf{s},\chi,\widetilde{\chi})\left(\int_Y\|u\tau_y\chi\|_{\mathcal{H}^{\mathbf{s}}}\langle z-y\rangle^{-N}\mathtt{d}\mu(y)\right)$$

Let $N=n+1$ and $1\leq p<\infty$. If (Z,υ) is either $\mathbb{R}^n$ with Lebesgue measure or a lattice with the counting measure, then Schur's lemma implies

$$\left(\int_Z\|u\tau_z\widetilde{\chi}\|_{\mathcal{H}^{\mathbf{s}}}^p\mathtt{d}\upsilon(z)\right)^{\frac{1}{p}}\leq C'(n,\mathbf{s},\chi,\widetilde{\chi})\left\|\langle\cdot\rangle^{-n-1}\right\|_{L^1}\left(\int_Y\|u\tau_y\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\mathtt{d}\mu(y)\right)^{\frac{1}{p}}$$

For $p=\infty$ we have

$$\sup_z\|u\tau_z\widetilde{\chi}\|_{\mathcal{H}^{\mathbf{s}}}\leq C'(n,\mathbf{s},\chi,\widetilde{\chi})\left\|\langle\cdot\rangle^{-n-1}\right\|_{L^1}\sup_y\|u\tau_y\chi\|_{\mathcal{H}^{\mathbf{s}}}^p.$$

By taking different combinations of (Y,μ) and (Z,υ) we obtain the following result.

Proposition 3.4. *Let $u\in\mathcal{D}'(\mathbb{R}^n)$ (or $u\in\mathcal{S}'(\mathbb{R}^n)$) and $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)\smallsetminus 0$ (or $\chi\in\mathcal{S}(\mathbb{R}^n)\smallsetminus 0$). Let $1\leq p<\infty$.*

(a) *If $\widetilde{\chi}\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\widetilde{\chi}\in\mathcal{S}(\mathbb{R}^n)$), then there is $C(n,\mathbf{s},\chi,\widetilde{\chi})>0$ such that*

$$\begin{aligned}\left(\int\|u\tau_{\widetilde{y}}\widetilde{\chi}\|_{\mathcal{H}^{\mathbf{s}}}\mathtt{d}\widetilde{y}\right)^{\frac{1}{p}} &\leq C(n,\mathbf{s},\chi,\widetilde{\chi})\left(\int\|u\tau_y\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\mathtt{d}y\right)^{\frac{1}{p}},\\ \sup_{\widetilde{y}}\|u\tau_{\widetilde{y}}\widetilde{\chi}\|_{\mathcal{H}^{\mathbf{s}}} &\leq C(n,\mathbf{s},\chi,\widetilde{\chi})\sup_y\|u\tau_y\chi\|_{\mathcal{H}^{\mathbf{s}}}.\end{aligned}$$

(b) *If* $\Gamma\subset\mathbb{R}^n$ *is a lattice such that*

$$\Psi=\Psi_{\Gamma,\chi}=\sum_{\gamma\in\Gamma}\left|\tau_\gamma\chi\right|^2>0$$

and $\widetilde{\chi}\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ *(or* $\widetilde{\chi}\in\mathcal{S}(\mathbb{R}^n)$*), then there is* $C(n,\mathbf{s},\Gamma,\chi,\widetilde{\chi})>0$ *such that*

$$\begin{aligned}\left(\int\left\|u\tau_{\widetilde{y}}\widetilde{\chi}\right\|_{\mathcal{H}^{\mathbf{s}}}\mathtt{d}\widetilde{y}\right)^{\frac{1}{p}} &\leq C(n,\mathbf{s},\Gamma,\chi,\widetilde{\chi})\left(\sum_{\gamma\in\Gamma}\left\|u\tau_\gamma\chi\right\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}},\\ \sup_{\widetilde{y}}\left\|u\tau_{\widetilde{y}}\widetilde{\chi}\right\|_{\mathcal{H}^{\mathbf{s}}} &\leq C(n,\mathbf{s},\Gamma,\chi,\widetilde{\chi})\sup_\gamma\left\|u\tau_\gamma\chi\right\|_{\mathcal{H}^{\mathbf{s}}}.\end{aligned}$$

(c) *If* $\widetilde{\Gamma}\subset\mathbb{R}^n$ *is a lattice and* $\widetilde{\chi}\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ *(or* $\widetilde{\chi}\in\mathcal{S}(\mathbb{R}^n)$*), then there is* $C\left(n,\mathbf{s},\widetilde{\Gamma},\chi,\widetilde{\chi}\right)>0$ *such that*

$$\begin{aligned}\left(\sum_{\widetilde{\gamma}\in\widetilde{\Gamma}}\left\|u\tau_{\widetilde{\gamma}}\chi\right\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} &\leq C\left(n,\mathbf{s},\widetilde{\Gamma},\chi,\widetilde{\chi}\right)\left(\int\left\|u\tau_y\chi\right\|_{\mathcal{H}^{\mathbf{s}}}^p\mathtt{d}y\right)^{\frac{1}{p}},\\ \sup_{\widetilde{\gamma}}\left\|u\tau_{\widetilde{\gamma}}\widetilde{\chi}\right\|_{\mathcal{H}^{\mathbf{s}}} &\leq C\left(n,\mathbf{s},\widetilde{\Gamma},\chi,\widetilde{\chi}\right)\sup_y\left\|u\tau_y\chi\right\|_{\mathcal{H}^{\mathbf{s}}}.\end{aligned}$$

(d) *If* $\Gamma,\widetilde{\Gamma}\subset\mathbb{R}^n$ *are lattices such that*

$$\Psi=\Psi_{\Gamma,\chi}=\sum_{\gamma\in\Gamma}\left|\tau_\gamma\chi\right|^2>0$$

and $\widetilde{\chi}\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ *(or* $\widetilde{\chi}\in\mathcal{S}(\mathbb{R}^n)$*), then there is* $C\left(n,\mathbf{s},\Gamma,\widetilde{\Gamma},\chi,\widetilde{\chi}\right)>0$ *such that*

$$\begin{aligned}\left(\sum_{\widetilde{\gamma}\in\widetilde{\Gamma}}\left\|u\tau_{\widetilde{\gamma}}\chi\right\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} &\leq C(n,\mathbf{s},\Gamma,\chi,\widetilde{\chi})\left(\sum_{\gamma\in\Gamma}\left\|u\tau_\gamma\chi\right\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}},\\ \sup_{\widetilde{\gamma}}\left\|u\tau_{\widetilde{\gamma}}\widetilde{\chi}\right\|_{\mathcal{H}^{\mathbf{s}}} &\leq C(n,\mathbf{s},\Gamma,\chi,\widetilde{\chi})\sup_\gamma\left\|u\tau_\gamma\chi\right\|_{\mathcal{H}^{\mathbf{s}}}.\end{aligned}$$

Definition 3.5. *Let* $1\leq p\leq\infty$, $\mathbf{s}\in\mathbb{R}^j$ *and* $u\in\mathcal{D}'(\mathbb{R}^n)$. *We say that* u *belongs to* $\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$ *if there is* $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)\smallsetminus 0$ *such that the measurable function* $\mathbb{R}^n\ni y\to\left\|u\tau_y\chi\right\|_{\mathcal{H}^{\mathbf{s}}}\in\mathbb{R}$ *belongs to* $L^p(\mathbb{R}^n)$. *We put*

$$\begin{aligned}\left\|u\right\|_{\mathbf{s},p,\chi} &= \left(\int\left\|u\tau_y\chi\right\|_{\mathcal{H}^{\mathbf{s}}}^p\mathtt{d}y\right)^{\frac{1}{p}},\quad 1\leq p<\infty,\\ \left\|u\right\|_{\mathbf{s},\infty,\chi} &\equiv \left\|u\right\|_{\mathbf{s},\mathtt{ul},\chi}=\sup_y\left\|u\tau_y\chi\right\|_{\mathcal{H}^{\mathbf{s}}}.\end{aligned}$$

Proposition 3.6. (a) *The above definition does not depend on the choice of the function* $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)\smallsetminus 0$.

(b) *If* $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)\smallsetminus 0$, *then* $\left\|\cdot\right\|_{\mathbf{s},p,\chi}$ *is a norm on* $\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$ *and the topology that defines does not depend on the function* χ.

(c) *Let* $\Gamma\subset\mathbb{R}^n$ *be a lattice and* $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ *be a function with the property that*

$$\Psi=\Psi_{\Gamma,\chi}=\sum_{\gamma\in\Gamma}\left|\tau_\gamma\chi\right|^2>0.$$

Then

$$\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n) \ni u \to \begin{cases} \left(\sum_{\gamma\in\Gamma} \|u\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} & 1 \le p < \infty \\ \sup_\gamma \|u\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}} & p = \infty \end{cases}$$

is a norm on $\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$ *and the topology that defines is the topology of* $\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$*. We shall use the notation*

$$\|u\|_{\mathbf{s},p,\Gamma,\chi} = \begin{cases} \left(\sum_{\gamma\in\Gamma} \|u\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} & 1 \le p < \infty \\ \sup_\gamma \|u\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}} & p = \infty \end{cases}.$$

(d) *If* $1 \le p \le q \le \infty$*, Then*

$$\mathcal{K}_1^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{K}_q^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{K}_\infty^{\mathbf{s}}(\mathbb{R}^n) \equiv \mathcal{H}_{\mathtt{ul}}^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

(e) *If* $s_1' \le s_1$*,...,*$s_j' \le s_j$*, then* $\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{K}_p^{\mathbf{s}'}(\mathbb{R}^n)$*.*

(f) $\left(\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n), \|\cdot\|_{\mathbf{s},p,\chi}\right)$ *is a Banach space.*

(g) $u \in \mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$ *if and only if there is* $l \in \{1,...,j\}$ *such that* $u, \partial_k u \in \mathcal{K}_p^{\mathbf{s}-\boldsymbol{\delta}_l}(\mathbb{R}^n)$ *for any* $k \in N_l$*, where* $\boldsymbol{\delta}_l = (\delta_{l1},...,\delta_{lj})$*.*

(h) *If* $s_1 > n_1/2, ..., s_j > n_j/2$*, then* $\mathcal{K}_\infty^{\mathbf{s}}(\mathbb{R}^n) \equiv \mathcal{H}_{\mathtt{ul}}^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n)$*.*

Proof. (a) (b) (c) are immediate consequences of the previous proposition.

(d) The inclusions $\mathcal{K}_1^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{K}_q^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{K}_\infty^{\mathbf{s}}(\mathbb{R}^n)$ are consequences of the elementary inclusions $l^1 \subset l^p \subset l^q \subset l^\infty$. What remains to be shown is the inclusion $\mathcal{K}_\infty^{\mathbf{s}}(\mathbb{R}^n) \equiv \mathcal{H}_{\mathtt{ul}}^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Let $u \in \mathcal{H}_{\mathtt{ul}}^{\mathbf{s}}(\mathbb{R}^n)$, $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \smallsetminus 0$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. We have

$$\begin{aligned} \langle u, \varphi\rangle &= \frac{1}{\|\chi\|_{L^2}^2} \int \langle u, (\tau_y\chi)(\tau_y\overline{\chi})\varphi\rangle\, \mathtt{d}y \\ &= \frac{1}{\|\chi\|_{L^2}^2} \int \langle u\tau_y\chi, (\tau_y\overline{\chi})\varphi\rangle\, \mathtt{d}y, \end{aligned}$$

$$\begin{aligned} |\langle u, \varphi\rangle| &\le \frac{1}{\|\chi\|_{L^2}^2} \int |\langle u\tau_y\chi, (\tau_y\overline{\chi})\varphi\rangle|\, \mathtt{d}y \\ &\le \frac{1}{\|\chi\|_{L^2}^2} \int \|u\tau_y\chi\|_{\mathcal{H}^{\mathbf{s}}} \|(\tau_y\overline{\chi})\varphi\|_{\mathcal{H}^{-\mathbf{s}}}\, \mathtt{d}y \\ &\le \frac{1}{\|\chi\|_{L^2}^2} \|u\|_{\mathbf{s},\infty,\chi} \int \|(\tau_y\overline{\chi})\varphi\|_{\mathcal{H}^{-\mathbf{s}}}\, \mathtt{d}y \end{aligned}$$

We shall use Proposition 2.4 to estimate $\|(\tau_y\overline{\chi})\varphi\|_{\mathcal{H}^{-\mathbf{s}}}$. Let $\widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\widetilde{\chi} = 1$ on $\mathtt{supp}\chi$. If $m_{\mathbf{s}} = \left[|\mathbf{s}|_1 + \frac{n+1}{2}\right] + 1$, then we obtain that

$$\begin{aligned} \|(\tau_y\overline{\chi})\varphi\|_{\mathcal{H}^{-\mathbf{s}}} &\le C \sup_{|\alpha+\beta|\le m_{\mathbf{s}}} |(\partial^\alpha\varphi)(\tau_y\partial^\alpha\overline{\chi})| \|\tau_y\widetilde{\chi}\|_{\mathcal{H}^{-\mathbf{s}}} \\ &= C \sup_{|\alpha+\beta|\le m_{\mathbf{s}}} |(\partial^\alpha\varphi)(\tau_y\partial^\alpha\overline{\chi})| \|\widetilde{\chi}\|_{\mathcal{H}^{-\mathbf{s}}}. \end{aligned}$$

Since χ, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ it follows that there is a continuous seminorm $p = p_{n,\mathbf{s}}$ on $\mathcal{S}(\mathbb{R}^n)$ so that

$$\begin{aligned}\left|(\partial^\alpha \varphi)\left(\tau_y \partial^\beta \overline{\chi}\right)(x)\right| &\leq p(\varphi)\, p(\chi)\, \langle x-y\rangle^{-2(n+1)} \langle x\rangle^{-2(n+1)} \\ &\leq 2^{n+1} p(\varphi)\, p(\chi)\, \langle 2x-y\rangle^{-(n+1)} \langle y\rangle^{-(n+1)} \\ &\leq 2^{n+1} p(\varphi)\, p(\chi)\, \langle y\rangle^{-(n+1)}, \quad |\alpha+\beta| \leq m_{\mathbf{s}}.\end{aligned}$$

Hence

$$|\langle u,\varphi\rangle| \leq 2^{n+1} C \frac{1}{\|\chi\|_{L^2}^2} \|u\|_{\mathbf{s},\infty,\chi} \left\|\langle\cdot\rangle^{-(n+1)}\right\|_{L^1} \|\widetilde{\chi}\|_{\mathcal{H}^{-\mathbf{s}}}\, p(\chi)\, p(\varphi).$$

(e) is trivial.

(f) Let $\{u_n\}$ be a Cauchy sequence in $\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$. Since $\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ is sequentially complete, there is $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $u_n \to u$ in $\mathcal{S}'(\mathbb{R}^n)$.

Let $\Gamma \subset \mathbb{R}^n$ be a lattice and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a function with the property that

$$\Psi = \Psi_{\Gamma,\chi} = \sum_{\gamma\in\Gamma} |\tau_\gamma \chi|^2 > 0.$$

Then for any $\gamma \in \Gamma$ there is $u_\gamma \in \mathcal{H}^{\mathbf{s}}$ such that $u_n \tau_\gamma \chi \to u_\gamma$ in $\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$. As $u_n \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ it follows that $u_\gamma = u\tau_\gamma\chi$ for any $\gamma \in \Gamma$.

Since $\{u_n\}$ is a Cauchy sequence in $\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$ there is $M \in (0,\infty)$ such that $\|u_n\|_{\mathbf{s},p,\Gamma,\chi} \leq M$ for any $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then there is n_ε such that if m, $n \geq n_\varepsilon$, then $\|u_m - u_n\|_{\mathbf{s},p,\Gamma,\chi} < \varepsilon$.

Let $F \subset \Gamma$ a finite subset. Then

$$\begin{aligned}\left(\sum_{\gamma\in F} \|u\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} &\leq \left(\sum_{\gamma\in F} \|u\tau_\gamma\chi - u_n\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} + \left(\sum_{\gamma\in F} \|u_n\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{\gamma\in F} \|u\tau_\gamma\chi - u_n\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} + M\end{aligned}$$

By passing to the limit we obtain $\left(\sum_{\gamma\in F} \|u\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} \leq M$ for any $F \subset \Gamma$ a finite subset. Hence $u \in \mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$.

For $F \subset \Gamma$ a finite subset and $m, n \geq n_\varepsilon$ we have

$$\begin{aligned}&\left(\sum_{\gamma\in F} \|u\tau_\gamma\chi - u_n\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} \\ &\quad\leq \left(\sum_{\gamma\in F} \|u\tau_\gamma\chi - u_m\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} + \left(\sum_{\gamma\in F} \|u_n\tau_\gamma\chi - u_m\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} \\ &\quad\leq \left(\sum_{\gamma\in F} \|u\tau_\gamma\chi - u_m\tau_\gamma\chi\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}} + \varepsilon\end{aligned}$$

By letting $m \to \infty$ we obtain $\left(\sum_{\gamma\in F}\|u\tau_\gamma\chi - u_n\tau_\gamma\chi\|^p_{\mathcal{H}^{\mathbf{s}}}\right)^{\frac{1}{p}} \le \varepsilon$ for any $F \subset \Gamma$ a finite subset and $n \ge n_\varepsilon$. This implies that $u_n \to u$ in $\mathcal{K}^{\mathbf{s}}_p(\mathbb{R}^n)$. The case $p = \infty$ is even simpler. □

Proposition 3.7. *Let* $\mathbf{s}$, $\mathbf{t}$, $\boldsymbol{\varepsilon}$, $\sigma(\boldsymbol{\varepsilon}) \in \mathbb{R}^j$ *such that,* $s_l + t_l > n_l/2$, $0 < \varepsilon_l < s_l + t_l - n_l/2$, $\sigma_l(\boldsymbol{\varepsilon}) = \sigma_l(\varepsilon_l) = \min\{s_l, t_l, s_l + t_l - n_l/2 - \varepsilon_l\}$ *for any* $l \in \{1, ..., j\}$. *If* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, *then*

$$\mathcal{K}^{\mathbf{s}}_p(\mathbb{R}^n) \cdot \mathcal{H}^{\mathbf{t}}_q(\mathbb{R}^n) \subset \mathcal{H}^{\boldsymbol{\sigma}}_r(\mathbb{R}^n)$$

Proof. Let $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^n) \smallsetminus 0$, $u \in \mathcal{K}^{\mathbf{s}}_p(\mathbb{R}^n)$ şi $v \in \mathcal{H}^{\mathbf{t}}_q(\mathbb{R}^n)$. By using Proposition 2.9 we obtain that $uv\tau_y\chi^2 \in \mathcal{H}^{\boldsymbol{\sigma}}$ and

$$\left\|uv\tau_y\chi^2\right\|_{\mathcal{H}^{\boldsymbol{\sigma}}} \le C\left\|u\tau_y\chi\right\|_{\mathcal{H}^{\mathbf{s}}}\left\|u\tau_y\chi\right\|_{\mathcal{H}^{\mathbf{t}}}$$

Finally, Hölder's inequality implies that

$$\|uv\|_{\boldsymbol{\sigma},r,\chi^2} \le C\|u\|_{\mathbf{s},p,\chi}\|v\|_{\mathbf{s},q,\chi}$$

□

Corollary 3.8. *Let* $\mathbf{s} \in \mathbb{R}^j$ *and* $1 \le p \le \infty$. *If* $s_1 > n_1/2, ..., s_j > n_j/2$, *then* $\mathcal{K}^{\mathbf{s}}_p(\mathbb{R}^n)$ *is an ideal in* $\mathcal{K}^{\mathbf{s}}_\infty(\mathbb{R}^n) \equiv \mathcal{H}^{\mathbf{s}}_{\mathtt{ul}}(\mathbb{R}^n)$ *with respect to the usual product.*

Now using the techniques of Coifman and Meyer, developed for the study of Beurling algebras A_ω and B_ω (see [Co-Me] pp 7-10), we shall prove an interesting result.

Theorem 3.9. $\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n) = \mathcal{K}^{\mathbf{s}}_2(\mathbb{R}^n)$.

To prove the result, we shall use partition of unity built in the previous section. Let $N \in \mathbb{N}$ and $\{x_1, ..., x_N\} \subset \mathbb{R}^n$ be such that

$$[0,1]^n \subset \left(x_1 + \left[\frac{1}{3}, \frac{2}{3}\right]^n\right) \cup ... \cup \left(x_N + \left[\frac{1}{3}, \frac{2}{3}\right]^n\right)$$

Let $\widetilde{h} \in \mathcal{C}^\infty_0(\mathbb{R}^n)$, $\widetilde{h} \ge 0$, be such that $\widetilde{h} = 1$ on $\left[\frac{1}{3}, \frac{2}{3}\right]^n$ and $\mathtt{supp}\widetilde{h} \subset \left[\frac{1}{4}, \frac{3}{4}\right]^n$. Then

(a) $\widetilde{H} = \sum_{i=1}^N \sum_{\gamma\in\mathbb{Z}^n} \tau_{\gamma+x_i}\widetilde{h} \in \mathcal{BC}^\infty(\mathbb{R}^n)$ is $\mathbb{Z}^n$-periodic and $\widetilde{H} \ge 1$.

(b) $h_i = \frac{\tau_{x_i}\widetilde{h}}{\widetilde{H}} \in \mathcal{C}^\infty_0(\mathbb{R}^n)$, $h_i \ge 0$, $\mathtt{supp}h_i \subset x_i + \left[\frac{1}{4}, \frac{3}{4}\right]^n = K_i$, $(K_i - K_i) \cap \mathbb{Z}^n = \{0\}$, $i = 1, ..., N$.

(c) $\chi_i = \sum_{\gamma\in\mathbb{Z}^n} \tau_\gamma h_i \in \mathcal{BC}^\infty(\mathbb{R}^n)$ is $\mathbb{Z}^n$-periodic, $i = 1, ..., N$ and $\sum_{i=1}^N \chi_i = 1$.

(d) $h = \sum_{i=1}^N h_i \in \mathcal{C}^\infty_0(\mathbb{R}^n)$, $h \ge 0$, $\sum_{\gamma\in\mathbb{Z}^n} \tau_\gamma h = 1$.

Lemma 3.10. $\mathcal{K}^{\mathbf{s}}_2(\mathbb{R}^n) \subset \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$.

Proof. Let $u \in \mathcal{K}^{\mathbf{s}}_2(\mathbb{R}^n)$. We have

$$u = \sum_{j=1}^N \chi_j u \quad \textit{with} \quad \chi_j u = \sum_{\gamma\in\mathbb{Z}^n} (\tau_\gamma h_j)\, u.$$

Since $u \in \mathcal{K}^{\mathbf{s}}_2(\mathbb{R}^n)$ applying Proposition 3.4 we get that

$$\sum_{\gamma\in\mathbb{Z}^n} \|(\tau_\gamma h_j)\, u\|^2_{\mathcal{H}^{\mathbf{s}}} < \infty.$$

Using Lemma 2.3 it follows that $\chi_j u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$ and

$$\left\|\chi_j u\right\|_{\mathcal{H}^{\mathbf{s}}} \approx \left(\sum_{\gamma\in\mathbb{Z}^n} \left\|(\tau_\gamma h_j)\, u\right\|^2_{\mathcal{H}^{\mathbf{s}}}\right)^{\frac{1}{2}} \leq C_j \left\|u\right\|_{\mathbf{s},2}$$

where $\left\|\cdot\right\|_{\mathbf{s},2}$ is a fixed norm on $\mathcal{K}^{\mathbf{s}}_2(\mathbb{R}^n)$. So $u = \sum_{j=1}^N \chi_j u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$ and

$$\left\|u\right\|_{\mathcal{H}^{\mathbf{s}}} \leq \sum_{j=1}^N \left\|\chi_j u\right\|_{\mathcal{H}^{\mathbf{s}}} \leq \left(\sum_{j=1}^N C_j\right) \left\|u\right\|_{\mathbf{s},2}.$$

□

Lemma 3.11. $\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n) \subset \mathcal{K}^{\mathbf{s}}_2(\mathbb{R}^n)$.

Proof. Then the following statements are equivalent:
(i) $u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$
(ii) $\chi_j u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$, $j = 1, ..., N$. (Here we use Proposition 2.1 (c))
(iii) $\left\{\left\|(\tau_\gamma h_j)\, u\right\|_{\mathcal{H}^{\mathbf{s}}}\right\}_{\gamma\in\mathbb{Z}^n} \in l^2(\mathbb{Z}^n)$, $j = 1, ..., N$. (Here we use Lemma 2.3)
Since $h = \sum_{j=1}^N h_j$ and

$$\left\|(\tau_\gamma h)\, u\right\|_{\mathcal{H}^{\mathbf{s}}} \leq \sum_{j=1}^N \left\|(\tau_\gamma h_j)\, u\right\|_{\mathcal{H}^{\mathbf{s}}}, \quad \gamma \in \mathbb{Z}^n$$

we get that $\left\{\left\|(\tau_\gamma h)\, u\right\|_{\mathcal{H}^{\mathbf{s}}}\right\}_{\gamma\in\mathbb{Z}^n} \in l^2(\mathbb{Z}^n)$. Since $h = \sum_{j=1}^N h_j \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $h \geq 0$, $\sum_{\gamma\in\mathbb{Z}^n} \tau_\gamma h = 1$ it follows that $u \in \mathcal{K}^{\mathbf{s}}_2(\mathbb{R}^n)$ and

$$\begin{aligned}
\left\|u\right\|_{\mathbf{s},2,h} &\approx \left\|\left\{\left\|(\tau_\gamma h)\, u\right\|_{\mathcal{H}^{\mathbf{s}}}\right\}_{\gamma\in\mathbb{Z}^n}\right\|_{l^2(\mathbb{Z}^n)} \\
&\leq \sum_{j=1}^N \left\|\left\{\left\|(\tau_\gamma h_j)\, u\right\|_{\mathcal{H}^{\mathbf{s}}}\right\}_{\gamma\in\mathbb{Z}^n}\right\|_{l^2(\mathbb{Z}^n)} \\
&\approx \sum_{j=1}^N \left\|\chi_j u\right\|_{\mathcal{H}^{\mathbf{s}}} \leq Cst \left\|u\right\|_{\mathcal{H}^{\mathbf{s}}}.
\end{aligned}$$

□

Corollary 3.12 (Kato)**.** *Let* $\mathbf{s}$, $\mathbf{t}$, $\boldsymbol{\varepsilon}$, $\sigma(\boldsymbol{\varepsilon}) \in \mathbb{R}^j$ *such that,* $s_l + t_l > n_l/2$, $0 < \varepsilon_l < s_l + t_l - n_l/2$, $\sigma_l(\boldsymbol{\varepsilon}) = \sigma_l(\varepsilon_l) = \min\{s_l, t_l, s_l + t_l - n_l/2 - \varepsilon_l\}$ *for any* $l \in \{1, ..., j\}$. *Then*

$$\mathcal{H}^{\mathbf{s}}_{\mathtt{ul}}(\mathbb{R}^n) \cdot \mathcal{H}^{\mathbf{t}}(\mathbb{R}^n) \subset \mathcal{H}^{\boldsymbol{\sigma}}(\mathbb{R}^n), \quad \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n) \cdot \mathcal{H}^{\mathbf{t}}_{\mathtt{ul}}(\mathbb{R}^n) \subset \mathcal{H}^{\boldsymbol{\sigma}}(\mathbb{R}^n).$$

Lemma 3.13. *If* $1 \leq p < \infty$, *then* $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $\mathcal{K}^{\mathbf{s}}_p(\mathbb{R}^n)$.

Proof. (i) Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\psi = 1$ on $B(0,1)$, $\psi^\varepsilon(x) = \psi(\varepsilon x)$, $0 < \varepsilon \leq 1$, $x \in \mathbb{R}^n$. If $u \in \mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$, then $\psi^\varepsilon u \to u$ in $\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)$. Moreover we have

$$\left\|\psi^\varepsilon u\right\|_{\mathcal{H}^{\mathbf{s}}} \leq C(s, n, \psi) \left\|u\right\|_{\mathcal{H}^{\mathbf{s}}}, \quad 0 < \varepsilon \leq 1,$$

where

$$\begin{aligned} C(s,n,\psi) &= (2\pi)^{-n} 2^{|\mathbf{s}|_1/2} \sup_{0<\varepsilon\le 1}\left(\int \langle \eta\rangle^{|\mathbf{s}|_1} \varepsilon^{-n}\left|\widehat{\psi}(\eta/\varepsilon)\right| \mathtt{d}\eta\right) \\ &= (2\pi)^{-n} 2^{|\mathbf{s}|_1/2} \sup_{0<\varepsilon\le 1}\left(\int \langle \varepsilon\eta\rangle^{|\mathbf{s}|_1} \left|\widehat{\psi}(\eta)\right| \mathtt{d}\eta\right) \\ &= (2\pi)^{-n} 2^{|\mathbf{s}|_1/2} \left(\int \langle \eta\rangle^{|\mathbf{s}|_1} \left|\widehat{\psi}(\eta)\right| \mathtt{d}\eta\right). \end{aligned}$$

`(ii)` Suppose that $u \in \mathcal{K}^{\mathbf{s}}_p(\mathbb{R}^n)$. Let $F \subset \mathbb{Z}^n$ be an arbitrary finite subset. Then the subadditivity property of the norm $\|\cdot\|_{l^p}$ implies that:

$$\begin{aligned} \|\psi^\varepsilon u - u\|_{\mathbf{s},p,\mathbb{Z}^n,\chi} &\le \left(\sum_{\gamma\in F}\|\psi^\varepsilon u\tau_\gamma\chi - u\tau_\gamma\chi\|^p_{\mathcal{H}^{\mathbf{s}}}\right)^{\frac{1}{p}} + \left(\sum_{\gamma\in\mathbb{Z}^n\smallsetminus F}\|\psi^\varepsilon u\tau_\gamma\chi\|^p_{\mathcal{H}^{\mathbf{s}}}\right)^{\frac{1}{p}} \\ &\quad + \left(\sum_{\gamma\in\mathbb{Z}^n\smallsetminus F}\|u\tau_\gamma\chi\|^p_{\mathcal{H}^{\mathbf{s}}}\right)^{\frac{1}{p}} \\ &\le \left(\sum_{\gamma\in F}\|\psi^\varepsilon u\tau_\gamma\chi - u\tau_\gamma\chi\|^p_{\mathcal{H}^{\mathbf{s}}}\right)^{\frac{1}{p}} + (C(s,n,\psi)+1)\left(\sum_{\gamma\in\mathbb{Z}^n\smallsetminus F}\|u\tau_\gamma\chi\|^p_{\mathcal{H}^{\mathbf{s}}}\right)^{\frac{1}{p}} \end{aligned}$$

By making $\varepsilon \to 0$ we deduce that

$$\limsup_{\varepsilon\to 0}\|\psi^\varepsilon u - u\|_{\mathbf{s},p,\mathbb{Z}^n,\chi} \le (C(s,n,\psi)+1)\left(\sum_{\gamma\in\mathbb{Z}^n\smallsetminus F}\|u\tau_\gamma\chi\|^p_{\mathcal{H}^{\mathbf{s}}}\right)^{\frac{1}{p}}$$

for any $F \subset \mathbb{Z}^n$ finite subset. Hence $\lim_{\varepsilon\to 0}\psi^\varepsilon u = u$ in $\mathcal{K}^{\mathbf{s}}_p(\mathbb{R}^n)$. The immediate consequence is that

`(iii)` $\mathcal{E}'(\mathbb{R}^n) \cap \mathcal{K}^{\mathbf{s}}_p(\mathbb{R}^n)$ is dense in $\mathcal{K}^{\mathbf{s}}_p(\mathbb{R}^n)$.

`(iv)` Suppose that $u \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{K}^{\mathbf{s}}_p(\mathbb{R}^n)$. Let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ be such that $\mathtt{supp}\varphi \subset \overline{B(0;1)}$, $\int \varphi(x)\,\mathtt{d}x = 1$. For $\varepsilon \in (0,1]$, we set $\varphi_\varepsilon = \varepsilon^{-n}\varphi(\cdot/\varepsilon)$. Let $K = \mathtt{supp}u + \overline{B(0;1)}$. Let $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^n) \smallsetminus 0$. Then there is a finite set $F = F_{K,\chi} \subset \mathbb{Z}^n$ such that $(\tau_\gamma\chi)(\varphi_\varepsilon * u - u) = 0$ for any $\gamma \in \mathbb{Z}^n \smallsetminus F$. It follows that

$$\begin{aligned} \|\varphi_\varepsilon * u - u\|_{\mathbf{s},p,\mathbb{Z}^n,\chi} &= \left(\sum_{\gamma\in F}\|(\tau_\gamma\chi)(\varphi_\varepsilon * u - u)\|^p_{\mathcal{H}^{\mathbf{s}}}\right)^{\frac{1}{p}} \\ &\approx \left(\sum_{\gamma\in F}\|(\tau_\gamma\chi)(\varphi_\varepsilon * u - u)\|^2_{\mathcal{H}^{\mathbf{s}}}\right)^{\frac{1}{2}} \\ &\approx \|\varphi_\varepsilon * u - u\|_{\mathcal{H}^{\mathbf{s}}} \to 0, \quad as\ \varepsilon \to 0. \end{aligned}$$

□

We end this section with an interpolation result. We choose $\chi_{\mathbb{Z}^n} \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ so that $\sum_{k\in\mathbb{Z}^n}\chi_{\mathbb{Z}^n}(\cdot - k) = 1$. For $k \in \mathbb{Z}^n$ we define the operator

$$S_k : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n), \quad S_k u = (\tau_k\chi_{\mathbb{Z}^n})\,u.$$

Now from the definition of $\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$ it follows that the linear operator

$$S:\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)\to l^p(\mathbb{Z}^n,\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n)),\quad Su=(S_ku)_{k\in\mathbb{Z}^n}$$

is well defined and continuous.

On the other hand, for any $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ the operator

$$R_\chi:l^p(\mathbb{Z}^n,\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n))\to\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n),$$
$$R_\chi\left((u_k)_{k\in\mathbb{Z}^n}\right)=\sum_{k\in\mathbb{Z}^n}(\tau_k\chi)u_k$$

is well defined and continuous.

Let $\underline{\mathbf{u}}=(u_k)_{k\in\mathbb{Z}^n}\in l^p(\mathbb{Z}^n,\mathcal{H}^{\mathbf{s}}(\mathbb{R}^n))$. Using Proposition 2.4 we get

$$\|(\tau_{k'}\chi_{\mathbb{Z}^n})(\tau_k\chi)u_k\|_{\mathcal{H}^{\mathbf{s}}}\leq Cst\sup_{|\alpha+\beta|\leq m_{\mathbf{s}}}\left|\left((\tau_{k'}\partial^\alpha\chi_{\mathbb{Z}^n})\left(\tau_k\partial^\beta\chi\right)\right)\right|\|u_k\|_{\mathcal{H}^{\mathbf{s}}}.$$

where $m_{\mathbf{s}}=\left[|\mathbf{s}|_1+\frac{n+1}{2}\right]+1$. Now for some continuous seminorm $p=p_{n,\mathbf{s}}$ on $\mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}\left|\left((\tau_{k'}\partial^\alpha\chi_{\mathbb{Z}^n})\left(\tau_k\partial^\beta\chi\right)\right)(x)\right| &\leq p(\chi_{\mathbb{Z}^n})p(\chi)\langle x-k'\rangle^{-2(n+1)}\langle x-k\rangle^{-2(n+1)}\\ &\leq 2^{n+1}p(\chi_{\mathbb{Z}^n})p(\chi)\langle 2x-k'-k\rangle^{-n-1}\langle k'-k\rangle^{-n-1}\\ &\leq 2^{n+1}p(\chi_{\mathbb{Z}^n})p(\chi)\langle k'-k\rangle^{-n-1},\quad |\alpha+\beta|\leq m_{\mathbf{s}}.\end{aligned}$$

Hence

$$\sup_{|\alpha+\beta|\leq m_{\mathbf{s}}}\left|\left((\tau_{k'}\partial^\alpha\chi_{\mathbb{Z}^n})\left(\tau_k\partial^\beta\chi\right)\right)\right|\leq 2^{n+1}p(\chi_{\mathbb{Z}^n})p(\chi)\langle k'-k\rangle^{-n-1},$$
$$\|(\tau_{k'}\chi_{\mathbb{Z}^n})(\tau_k\chi)u_k\|_{\mathcal{H}^{\mathbf{s}}}\leq C(n,\mathbf{s},\chi_{\mathbb{Z}^n},\chi)\langle k'-k\rangle^{-n-1}\|u_k\|_{\mathcal{H}^{\mathbf{s}}}.$$

The last estimate implies that

$$\|(\tau_{k'}\chi_{\mathbb{Z}^n})R_\chi(\underline{\mathbf{u}})\|_{\mathcal{H}^{\mathbf{s}}}\leq C(n,\mathbf{s},\chi_{\mathbb{Z}^n},\chi)\sum_{k\in\mathbb{Z}^n}\langle k'-k\rangle^{-n-1}\|u_k\|_{\mathcal{H}^{\mathbf{s}}}.$$

Now Schur's lemma implies the result

$$\left(\sum_{k'\in\mathbb{Z}^n}\|(\tau_{k'}\chi_{\mathbb{Z}^n})R_\chi(\underline{\mathbf{u}})\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}}\leq C'(n,\mathbf{s},\chi_{\mathbb{Z}^n},\chi)\left\|\langle\cdot\rangle^{-n-1}\right\|_{L^1}\left(\sum_{k\in\mathbb{Z}^n}\|u_k\|_{\mathcal{H}^{\mathbf{s}}}^p\right)^{\frac{1}{p}}.$$

If $\chi=1$ on a neighborhood of $\mathtt{supp}\chi_{\mathbb{Z}^n}$, then $\chi\chi_{\mathbb{Z}^n}=\chi_{\mathbb{Z}^n}$ and as a consequence $R_\chi S=\mathtt{Id}_{S_w^p(\mathbb{R}^n)}$:

$$\begin{aligned}R_\chi Su &= \sum_{k\in\mathbb{Z}^n}(\tau_k\chi)S_ku=\sum_{k\in\mathbb{Z}^n}(\tau_k\chi)(\tau_k\chi_{\mathbb{Z}^n})u\\ &= \sum_{k\in\mathbb{Z}^n}(\tau_k\chi_{\mathbb{Z}^n})u=u.\end{aligned}$$

Thus we proved the following result.

Proposition 3.14. *Under the above conditions, the operator* $R_\chi:l^p(\mathbb{Z}^n,\mathcal{H}^{\mathbf{s}})\to\mathcal{K}_p^{\mathbf{s}}$ *is a retract.*

Using the results of [Tri] section 1.18 we obtain the following corollary.

Corollary 3.15. *For* $0<\theta<1$

$$\mathcal{K}_{\frac{1}{1-\theta}}^{\mathbf{s}}(\mathbb{R}^n)=\left[\mathcal{K}_1^{\mathbf{s}}(\mathbb{R}^n),\mathcal{K}_\infty^{\mathbf{s}}(\mathbb{R}^n)\right]_\theta$$

4. Wiener-Lévy theorem for Kato-Sobolev algebras

We shall work only in the case $j = 1$, i.e. only in the case of the usual Kato-Sobolev spaces. The case $j > 1$ can be treated in the same way but with more complicated notations and statements which can hide the ideas and the beauty of some arguments. So

$$\mathcal{H}^s(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n) : (1 - \triangle_{\mathbb{R}^n})^{s/2} u \in L^2(\mathbb{R}^n)\right\},$$
$$\|u\|_{\mathcal{H}^{\mathbf{s}}} = \left\|(1 - \triangle_{\mathbb{R}^n})^{s/2} u\right\|_{L^2}, \quad u \in \mathcal{H}^{\mathbf{s}},$$

Let $1 \le p \le \infty$, $s \in \mathbb{R}$ and $u \in \mathcal{D}'(\mathbb{R}^n)$. We say that u belongs to $u \in \mathcal{K}^s_p(\mathbb{R}^n)$ if there is $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^n) \diagdown 0$ such that the measurable function $\mathbb{R}^n \ni y \to \|u\tau_y\chi\|_{\mathcal{H}^s} \in \mathbb{R}$ belongs to $L^p(\mathbb{R}^n)$. We put

$$\begin{aligned} \|u\|_{s,p,\chi} &= \left(\int \|u\tau_y\chi\|^p_{\mathcal{H}^s} \mathtt{d}y\right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \\ \|u\|_{s,\infty,\chi} &\equiv \|u\|_{s,\mathtt{ul},\chi} = \sup_y \|u\tau_y\chi\|_{\mathcal{H}^s}. \end{aligned}$$

In Kato's notation $\mathcal{K}^s_\infty(\mathbb{R}^n) \equiv \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$ the uniformly local Sobolev space of order s.

Lemma 4.1. (a) *$\mathcal{BC}^m(\mathbb{R}^n) \subset \mathcal{H}^m_{\mathtt{ul}}(\mathbb{R}^n)$ for any $m \in \mathbb{N}$.*
(b) *$\mathcal{BC}^{[|s|]+1}(\mathbb{R}^n) \subset \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$ for any $s \in \mathbb{R}$.*

Proof. (a) Let $u \in \mathcal{BC}^m(\mathbb{R}^n)$ and $\chi \in \mathcal{S}(\mathbb{R}^n)$. Then using Leibniz's formula

$$\partial^\alpha(u\tau_y\chi) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta u \cdot \tau_y \partial^{\alpha-\beta}\chi$$

we get that $\partial^\alpha(u\tau_y\chi) \in L^2(\mathbb{R}^n)$ for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \le m$. Also there is $C = C(m,n) > 0$ such that

$$\|u\tau_y\chi\|_{\mathcal{H}^m} \approx \left(\sum_{|\alpha| \le m} \|\partial^\alpha(u\tau_y\chi)\|^2_{L^2}\right)^{1/2} \le C\|u\|_{\mathcal{BC}^m}\|\chi\|_{\mathcal{H}^m}, \quad y \in \mathbb{R}^n$$

which implies

$$\|u\|_{m,\mathtt{ul},\chi} \le C\|u\|_{\mathcal{BC}^m}\|\chi\|_{\mathcal{H}^m}.$$

(b) We have $\mathcal{BC}^{[|s|]+1}(\mathbb{R}^n) \subset \mathcal{H}^{[|s|]+1}_{\mathtt{ul}}(\mathbb{R}^n) \subset \mathcal{H}^{|s|}_{\mathtt{ul}}(\mathbb{R}^n) \subset \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$. □

Let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^n)$, $\varphi \ge 0$ be such that $\mathtt{supp}\varphi \subset B(0;1)$, $\int \varphi(x)\,\mathtt{d}x = 1$. For $\varepsilon \in (0,1]$, we set $\varphi_\varepsilon = \varepsilon^{-n}\varphi(\cdot/\varepsilon)$.

Lemma 4.2. *If $s' \le s$, then*

$$\|\varphi_\varepsilon * u - u\|_{\mathcal{H}^{s'}} \le 2^{1-\min\{s-s',1\}}\varepsilon^{\min\{s-s',1\}}\|u\|_{\mathcal{H}^s}, \quad u \in \mathcal{H}^s(\mathbb{R}^n).$$

Proof. We have

$$\mathcal{F}(\varphi_\varepsilon * u - u)(\xi) = (\widehat{\varphi}(\varepsilon\xi) - 1)\,\widehat{u}(\xi)$$

with

$$\widehat{\varphi}(\varepsilon\xi) - 1 = \int \left(\mathtt{e}^{-\mathtt{i}\langle x,\varepsilon\xi\rangle} - 1\right)\varphi(x)\,\mathtt{d}x$$

Since $\left|\mathtt{e}^{-\mathtt{i}\lambda}-1\right| \leq |\lambda|$ we get

$$|\widehat{\varphi}(\varepsilon\xi)-1| \leq \left\{ \begin{array}{c} 2\int\varphi(x)\,\mathtt{d}x \\ \varepsilon|\xi|\int|x|\,\varphi(x)\,\mathtt{d}x \end{array} \right. \leq \left\{ \begin{array}{c} 2 \\ \varepsilon|\xi| \end{array} \right.$$

If $0 \leq s-s' \leq 1$, then

$$\begin{aligned} |\widehat{\varphi}(\varepsilon\xi)-1| &= |\widehat{\varphi}(\varepsilon\xi)-1|^{1-\left(s-s'\right)} |\widehat{\varphi}(\varepsilon\xi)-1|^{s-s'} \\ &\leq 2^{1-\left(s-s'\right)}\varepsilon^{s-s'}|\xi|^{s-s'} \leq 2^{1-\left(s-s'\right)}\varepsilon^{s-s'}\langle\xi\rangle^{s-s'} \end{aligned}$$

which implies that

$$\|\varphi_\varepsilon * u - u\|_{\mathcal{H}^{s'}} \leq 2^{1-\left(s-s'\right)}\varepsilon^{s-s'}\|u\|_{\mathcal{H}^s}, \quad u \in \mathcal{H}^s(\mathbb{R}^n).$$

If $s' \leq s-1$, then

$$\|\varphi_\varepsilon * u - u\|_{\mathcal{H}^{s'}} \leq \varepsilon\|u\|_{\mathcal{H}^{s'+1}} \leq \varepsilon\|u\|_{\mathcal{H}^s}, \quad u \in \mathcal{H}^s(\mathbb{R}^n).$$

□

Let $\chi, \chi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^n) \smallsetminus 0$ be such that $\chi_0 = 1$ on $\mathtt{supp}\chi + B(0;1)$. Let $u \in \mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n)$. Then for $0 < \varepsilon \leq 1$ we have

$$\tau_y\chi(\varphi_\varepsilon * u - u) = \tau_y\chi(\varphi_\varepsilon * (u\tau_y\chi_0) - u\tau_y\chi_0).$$

Proposition 2.1 and the previous lemma imply

$$\begin{aligned} \|\tau_y\chi(\varphi_\varepsilon * u - u)\|_{\mathcal{H}^{s'}} &\leq C_{s',\chi}\|\varphi_\varepsilon * (u\tau_y\chi_0) - u\tau_y\chi_0\|_{\mathcal{H}^{s'}} \\ &\leq C_{s',\chi}2^{1-\min\left\{s-s',1\right\}}\varepsilon^{\min\left\{s-s',1\right\}}\|u\tau_y\chi_0\|_{\mathcal{H}^s} \end{aligned}$$

It follows that

$$\|\varphi_\varepsilon * u - u\|_{s',\mathtt{ul},\chi} \leq C_{s',\chi}2^{1-\min\left\{s-s',1\right\}}\varepsilon^{\min\left\{s-s',1\right\}}\|u\|_{s,\mathtt{ul},\chi_0}$$

Definition 4.3. $\mathcal{H}_{\mathtt{ul}}^{s\left(s'\right)}(\mathbb{R}^n) \equiv \left(\mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n), \|\cdot\|_{s',\mathtt{ul}}\right)$.

Corollary 4.4. (a) *If $s' < s$, then $\mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$ is dense in $\mathcal{H}_{\mathtt{ul}}^{s\left(s'\right)}(\mathbb{R}^n)$.*

(b) *If $\frac{n}{2} < s' < s$, then $\mathcal{BC}^\infty(\mathbb{R}^n)$ is dense in $\mathcal{H}_{\mathtt{ul}}^{s\left(s'\right)}(\mathbb{R}^n)$.*

Proof. (b) If $s > \frac{n}{2}$, then $\mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n)$. Therefore $\varphi_\varepsilon * \mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n) \subset \mathcal{BC}^\infty(\mathbb{R}^n)$. □

We need another auxiliary result.

Lemma 4.5. *The map*

$$\mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n) \ni (\varphi, u) \to \varphi * u \in \mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n)$$

is well defined and for any $\chi \in \mathcal{S}(\mathbb{R}^n) \smallsetminus 0$ we have the estimate

$$\|\varphi * u\|_{s,\mathtt{ul},\chi} \leq \|\varphi\|_{L^1}\|u\|_{s,\mathtt{ul},\chi}, \quad (\varphi, u) \in \mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n).$$

Proof. Let $(\varphi, u) \in \mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n)$, $\chi \in \mathcal{S}(\mathbb{R}^n) \smallsetminus 0$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then using (3.2) we obtain

$$\begin{aligned} \langle\tau_z\chi(\varphi * u), \psi\rangle &= \langle u, \check{\varphi} * ((\tau_z\chi)\psi)\rangle = \int\check{\varphi}(y)\langle u, \tau_y((\tau_z\chi)\psi)\rangle\,\mathtt{d}y \\ &= \int\varphi(y)\langle u, \tau_{-y}((\tau_z\chi)\psi)\rangle\,\mathtt{d}y = \int\varphi(y)\langle(\tau_{z-y}\chi)u, \tau_{-y}\psi\rangle\,\mathtt{d}y, \end{aligned}$$

where $\check{\varphi}(y) = \varphi(-y)$. Since

$$|\langle(\tau_{z-y}\chi)\, u, \tau_{-y}\psi\rangle| \leq \|(\tau_{z-y}\chi)\, u\|_{\mathcal{H}^s} \|\tau_{-y}\psi\|_{\mathcal{H}^{-s}} \leq \|u\|_{s,\mathtt{ul},\chi} \|\psi\|_{\mathcal{H}^{-s}}$$

it follows that

$$|\langle\tau_z\chi\,(\varphi * u)\,, \psi\rangle| \leq \|\varphi\|_{L^1} \|u\|_{s,\mathtt{ul},\chi} \|\psi\|_{\mathcal{H}^{-s}}$$

Hence $\tau_z\chi\,(\varphi * u) \in \mathcal{H}^s(\mathbb{R}^n)$ and $\|\tau_z\chi\,(\varphi * u)\|_{\mathcal{H}^s} \leq \|\varphi\|_{L^1} \|u\|_{s,\mathtt{ul},\chi}$ for every $z \in \mathbb{R}^n$, i.e. $\varphi * u \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$ and

$$\|\varphi * u\|_{s,\mathtt{ul},\chi} \leq \|\varphi\|_{L^1} \|u\|_{s,\mathtt{ul},\chi}$$

□

Theorem 4.6 (Wiener-Lévy for $\mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$, weak form)**.** *Let $\Omega = \mathring{\Omega} \subset \mathbb{C}^d$ and $\Phi : \Omega \to \mathbb{C}$ a holomorphic function. Let $s > n/2$.*

(a) *If $u = (u_1, ..., u_d) \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)^d$ satisfies the condition $\overline{u(\mathbb{R}^n)} \subset \Omega$, then*

$$\Phi \circ u \equiv \Phi(u) \in \mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n)\,, \quad \forall s' < s.$$

(b) *Suppose that $s' \in (n/2, s)$. If $u, u_\varepsilon \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)^d$, $0 < \varepsilon \leq 1$, $\overline{u(\mathbb{R}^n)} \subset \Omega$ and $u_\varepsilon \to u$ in $\mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n)^d$ as $\varepsilon \to 0$, then there is $\varepsilon_0 \in (0, 1]$ such that $\overline{u_\varepsilon(\mathbb{R}^n)} \subset \Omega$ for every $0 < \varepsilon \leq \varepsilon_0$ and $\Phi(u_\varepsilon) \to \Phi(u)$ in $\mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n)$ as $\varepsilon \to 0$.*

Proof. On $\mathbb{C}^d$ we shall consider the distance given by the norm

$$|z|_\infty = \max\{|z_1|, ..., |z_d|\}\,, \quad z \in \mathbb{C}^d.$$

Let $r = \mathtt{dist}\left(\overline{u(\mathbb{R}^n)}, \mathbb{C}^d \smallsetminus \Omega\right)/8$. Since $\overline{u(\mathbb{R}^n)} \subset \Omega$ it follows that $r > 0$ and

$$\bigcup_{y \in \overline{u(\mathbb{R}^n)}} \overline{B(y; 4r)} \subset \Omega.$$

Let $s' \in (n/2, s)$. On $\mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n)^d$ we shall consider the norm

$$|||u|||_{s',\mathtt{ul}} = \max\left\{\|u_1\|_{s',\mathtt{ul}}, ..., \|u_d\|_{s',\mathtt{ul}}\right\}, \quad u \in \mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n)^d,$$

where $\|\cdot\|_{s',\mathtt{ul}}$ is a fixed Banach algebra norm on $\mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n)$, and on $\mathcal{BC}(\mathbb{R}^n)^d$ we shall consider the norm

$$|||u|||_\infty = \max\{\|u_1\|_\infty, ..., \|u_d\|_\infty\}\,, \quad u \in \mathcal{BC}(\mathbb{R}^n)^d.$$

Since $\mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n)$ there is $C \geq 1$ so that

$$\|\cdot\|_\infty \leq C\,\|\cdot\|_{s',\mathtt{ul}}$$

According to Corollary 4.4 $\mathcal{BC}^\infty(\mathbb{R}^n)$ is dense in $\mathcal{H}^{s(s')}_{\mathtt{ul}}(\mathbb{R}^n)$. Therefore we find $v = (v_1, ..., v_d) \in \mathcal{BC}^\infty(\mathbb{R}^n)^d$ so that

$$|||u - v|||_{s',\mathtt{ul}} < r/C.$$

Then

$$|||u - v|||_\infty \leq C\,|||u - v|||_{s',\mathtt{ul}} < r.$$

Using the last estimate we show that $\overline{v(\mathbb{R}^n)} \subset \bigcup_{x \in \mathbb{R}^n} B(u(x); r)$. Indeed, if $z \in \overline{v(\mathbb{R}^n)}$, then there is $x \in \mathbb{R}^n$ such that

$$|z - v(x)|_\infty < r - |||v - u|||_\infty$$

It follows that

$$\begin{aligned}\left|z-u\left(x\right)\right|_{\infty} &\leq \left|z-v\left(x\right)\right|_{\infty}+\left|v\left(x\right)-u\left(x\right)\right|_{\infty}\\ &\leq \left|z-v\left(x\right)\right|_{\infty}+\left|\left|\left|v-u\right|\right|\right|_{\infty}\\ &< r-\left|\left|\left|v-u\right|\right|\right|_{\infty}+\left|\left|\left|v-u\right|\right|\right|_{\infty}=r\end{aligned}$$

so $z\in B\left(u\left(x\right);r\right)$.

From $\overline{v\left(\mathbb{R}^{n}\right)}\subset\bigcup_{x\in\mathbb{R}^{n}}B\left(u\left(x\right);r\right)$ we get

$$\overline{v\left(\mathbb{R}^{n}\right)}+\overline{B\left(0;3r\right)}\subset\bigcup_{x\in\mathbb{R}^{n}}B\left(u\left(x\right);4r\right)\subset\Omega,$$

hence the map

$$\mathbb{R}^{n}\times\overline{B\left(0;3r\right)}\ni\left(x,\zeta\right)\rightarrow\Phi\left(v\left(x\right)+\zeta\right)\in\mathbb{C}.$$

is well defined. Let $\Gamma\left(r\right)$ denote the polydisc $\left(\partial\mathbb{D}\left(0,3r\right)\right)^{d}$. Since $\overline{v\left(\mathbb{R}^{n}\right)}+\Gamma\left(r\right)\subset\Omega$ is a compact subset, the map

$$\Gamma\left(r\right)\ni\zeta\rightarrow\Phi\left(\zeta+v\right)\in\mathcal{BC}^{\left[s'\right]+1}\left(\mathbb{R}^{n}\right)\subset\mathcal{H}_{\mathtt{ul}}^{s'}\left(\mathbb{R}^{n}\right)$$

is continuous.

On the other hand we have

$$\left(\zeta_{1}+v_{1}-u_{1}\right)^{-1},...,\left(\zeta_{d}+v_{d}-u_{d}\right)^{-1}\in\mathcal{H}_{\mathtt{ul}}^{s'}\left(\mathbb{R}^{n}\right)$$

because $\left\|u_{1}-v_{1}\right\|_{s',\mathtt{ul}},...,\left\|u_{d}-v_{d}\right\|_{s',\mathtt{ul}}<r/C\leq r$ and $\left|\zeta_{1}\right|=...=\left|\zeta_{d}\right|=3r$.

It follows that the integral

$$h=\frac{1}{\left(2\pi\mathtt{i}\right)^{d}}\int_{\Gamma\left(r\right)}\frac{\Phi\left(\zeta+v\right)}{\left(\zeta_{1}+v_{1}-u_{1}\right)...\left(\zeta_{d}+v_{d}-u_{d}\right)}\mathtt{d}\zeta \tag{4.1}$$

defines an element $h\in\mathcal{H}_{\mathtt{ul}}^{s'}\left(\mathbb{R}^{n}\right)$.

Let

$$\delta_{x}:\mathcal{H}_{\mathtt{ul}}^{s'}\left(\mathbb{R}^{n}\right)\subset\mathcal{BC}\left(\mathbb{R}^{n}\right)\rightarrow\mathbb{C},\quad w\rightarrow w\left(x\right),$$

be the evaluation functional at $x\in\mathbb{R}^{n}$. Then

$$\begin{aligned}h\left(x\right) &= \frac{1}{\left(2\pi\mathtt{i}\right)^{d}}\int_{\Gamma\left(r\right)}\frac{\Phi\left(\zeta+v\left(x\right)\right)}{\left(\zeta_{1}-\left(u_{1}\left(x\right)-v_{1}\left(x\right)\right)\right)...\left(\zeta_{d}-\left(u_{d}\left(x\right)-v_{d}\left(x\right)\right)\right)}\mathtt{d}\zeta\\ &= \Phi\left(\zeta+v\left(x\right)\right)|_{\zeta=u\left(x\right)-v\left(x\right)}=\Phi\left(u\left(x\right)\right)\end{aligned}$$

because $\left|u\left(x\right)-v\left(x\right)\right|_{\infty}\leq\left|\left|\left|u-v\right|\right|\right|_{\infty}<r$, so $u\left(x\right)-v\left(x\right)$ is within polydisc $\Gamma\left(r\right)$. Hence $h=\Phi\circ u\equiv\Phi\left(u\right)\in\mathcal{H}_{\mathtt{ul}}^{s'}\left(\mathbb{R}^{n}\right)$, for any $s'\in\left(n/2,s\right)$ so

$$\Phi\circ u\equiv\Phi\left(u\right)\in\mathcal{H}_{\mathtt{ul}}^{s'}\left(\mathbb{R}^{n}\right),\quad\forall s'<s.$$

($\mathtt{b}$) Let $\varepsilon_{0}\in\left(0,1\right]$ be such that for any $0<\varepsilon\leq\varepsilon_{0}$ we have

$$\left|\left|\left|u-u_{\varepsilon}\right|\right|\right|_{s',\mathtt{ul}}<r/C.$$

Then $\left|\left|\left|u-u_{\varepsilon}\right|\right|\right|_{\infty}\leq C\left|\left|\left|u-u_{\varepsilon}\right|\right|\right|_{s',\mathtt{ul}}<r$ and $\overline{u_{\varepsilon}\left(\mathbb{R}^{n}\right)}\subset\bigcup_{x\in\mathbb{R}^{n}}B\left(u\left(x\right);r\right)\subset\Omega$ for every $0<\varepsilon\leq\varepsilon_{0}$.

On the other hand we have $\left|\left|\left|v-u_{\varepsilon}\right|\right|\right|_{s',\mathtt{ul}}\leq\left|\left|\left|v-u\right|\right|\right|_{s',\mathtt{ul}}+\left|\left|\left|u-u_{\varepsilon}\right|\right|\right|_{s',\mathtt{ul}}<r/C+r/C\leq2r$. It follows that

$$\left(\zeta_{1}+v_{1}-u_{\varepsilon1}\right)^{-1},...,\left(\zeta_{d}+v_{d}-u_{\varepsilon d}\right)^{-1}\in\mathcal{H}_{\mathtt{ul}}^{s'}\left(\mathbb{R}^{n}\right)$$

because $\left\|u_{\varepsilon1}-v_{1}\right\|_{s',\mathtt{ul}},...,\left\|u_{\varepsilon d}-v_{d}\right\|_{s',\mathtt{ul}}<2r$ and $\left|\zeta_{1}\right|=...=\left|\zeta_{d}\right|=3r$.

We obtain that

$$\begin{aligned}\Phi(u_\varepsilon) &= \frac{1}{(2\pi\mathtt{i})^d}\int_{\Gamma(r)} \frac{\Phi(\zeta+v)}{(\zeta_1+v_1-u_{\varepsilon 1})\dots(\zeta_d+v_d-u_{\varepsilon d})}\mathtt{d}\zeta \\ &\to \frac{1}{(2\pi\mathtt{i})^d}\int_{\Gamma(r)} \frac{\Phi(\zeta+v)}{(\zeta_1+v_1-u_1)\dots(\zeta_d+v_d-u_d)}\mathtt{d}\zeta = \Phi(u)\end{aligned}$$

as $\varepsilon\to 0$. □

Remark 4.7. *According to Coquand and Stolzenberg* [CS]*, this type of representation formula, (4.1), was introduced more than 60 years ago by A. P. Calderón.*

Lemma 4.8. *Suppose that $s>\max\{n/2,3/4\}$. Let $\Omega=\mathring{\Omega}\subset\mathbb{C}^d$ and $\Phi:\Omega\to\mathbb{C}$ a holomorphic function. If $u=(u_1,...,u_d)\in\mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)^d$ satisfies the condition $\overline{u(\mathbb{R}^n)}\subset\Omega$, then*

$$\partial_j\Phi(u)=\sum_{k=1}^d\frac{\partial\Phi}{\partial z_k}(u)\cdot\partial_j u_k,\quad \textit{in } \mathcal{D}'(\mathbb{R}^n),\quad j=1,...,n.$$

Proof. Let s' be such that $\max\{n/2,3/4,s-1\}<s'<s$. Then $s'+s'-1>n/2$. Let $u=(u_1,...,u_d)\in(\mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n))^d$. We consider the family $\{u_\varepsilon\}_{0<\varepsilon\le 1}$

$$u_\varepsilon=\varphi_\varepsilon*u=(\varphi_\varepsilon*u_1,...,\varphi_\varepsilon*u_d)\in\mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)^d$$

Then $u_\varepsilon\to u$ in $\mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n)^d$ as $\varepsilon\to0$, $\partial_j u_\varepsilon=\varphi_\varepsilon*\partial_j u\in\mathcal{H}^{s-1}_{\mathtt{ul}}(\mathbb{R}^n)^d$ and $\partial_j u_\varepsilon\to\partial_j u$ in $\mathcal{H}^{s'-1}_{\mathtt{ul}}(\mathbb{R}^n)^d$ as $\varepsilon\to0$. Since $\Phi(u_\varepsilon)\to\Phi(u)$ in $\mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n)\subset\mathcal{BC}(\mathbb{R}^n)\subset\mathcal{D}'(\mathbb{R}^n)$ (Theorem 4.6 ($\mathtt{b}$)), it follows that $\partial_j\Phi(u_\varepsilon)\to\partial_j\Phi(u)$ in $\mathcal{D}'(\mathbb{R}^n)$, $j=1,...,n$.

On the other hand we have

$$\partial_j\Phi(u_\varepsilon)=\sum_{k=1}^d\frac{\partial\Phi}{\partial z_k}(u_\varepsilon)\cdot\partial_j u_{\varepsilon k},\quad \textit{in } \mathcal{C}^\infty(\mathbb{R}^n),\quad j=1,...,n.$$

Let $\delta>0$ be such that $s'-n/2-\delta>0$. Then

$$s'-1=\min\{s',s'-1,s'+s'-1-n/2-\delta\}.$$

Since

$$\frac{\partial\Phi}{\partial z_k}(u_\varepsilon)\to\frac{\partial\Phi}{\partial z_k}(u),\quad \textit{in } \mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n),\ (\text{Theorem 4.6}\,(\mathtt{b})),\quad k=1,...,d,$$
$$\partial_j u_\varepsilon\to\partial_j u,\quad \textit{in } \mathcal{H}^{s'-1}_{\mathtt{ul}}(\mathbb{R}^n)^d,\quad j=1,...,n,$$

using Proposition 3.7 we get that

$$\partial_j\Phi(u_\varepsilon)=\sum_{k=1}^d\frac{\partial\Phi}{\partial z_k}(u_\varepsilon)\cdot\partial_j u_{\varepsilon k}\to\partial_j\Phi(u)=\sum_{k=1}^d\frac{\partial\Phi}{\partial z_k}(u)\cdot\partial_j u_k,\quad \textit{in } \mathcal{H}^{s'-1}_{\mathtt{ul}}(\mathbb{R}^n)$$

for $j=1,...,n$. Hence

$$\partial_j\Phi(u)=\sum_{k=1}^d\frac{\partial\Phi}{\partial z_k}(u)\cdot\partial_j u_k,\quad \text{in } \mathcal{D}'(\mathbb{R}^n),\quad j=1,...,n.$$

□

Remark 4.9. *Let us note that $\partial_j u_{\varepsilon k}=\varphi_\varepsilon*\partial_j u_k\to\partial_j u_k$ in $\mathcal{H}^{s'-1}_{\mathtt{ul}}(\mathbb{R}^n)$, but $\partial_j u_k\in\mathcal{H}^{s-1}_{\mathtt{ul}}(\mathbb{R}^n)$, $j=1,...,n$, $k=1,...,d$. This remark leads to the complete version of the Wiener-Lévy theorem.*

Theorem 4.10 (Wiener-Lévy for $\mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$)**.** *Suppose that $s > \max\{n/2, 3/4\}$. Let $\Omega = \mathring{\Omega} \subset \mathbb{C}^d$ and $\Phi : \Omega \to \mathbb{C}$ a holomorphic function. If $u = (u_1, ..., u_d) \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)^d$ satisfies the condition $\overline{u(\mathbb{R}^n)} \subset \Omega$, then*

$$\Phi \circ u \equiv \Phi(u) \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n).$$

Proof. Let s' be such that $\max\{n/2, 3/4, s-1\} < s' < s$. Then $s' + s' - 1 > n/2$. Let $\delta > 0$ be such that $s' - n/2 - \delta > 0$. Then

$$s - 1 = \min\{s', s-1, s'+s-1-n/2-\delta\}.$$

Since

$$\frac{\partial \Phi}{\partial z_k}(u) \in \mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n), \text{ (Theorem 4.6 (a))}, \quad k = 1, ..., d,$$

$$\partial_j u \in \mathcal{H}^{s-1}_{\mathtt{ul}}(\mathbb{R}^n)^d, \quad j = 1, ..., n,$$

using Proposition 3.7 we get that

$$\partial_j \Phi(u) = \sum_{k=1}^{d} \frac{\partial \Phi}{\partial z_k}(u) \cdot \partial_j u_k \in \mathcal{H}^{s-1}_{\mathtt{ul}}(\mathbb{R}^n), \quad j = 1, ..., n.$$

Now $\Phi(u) \in \mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n) \subset \mathcal{H}^{s-1}_{\mathtt{ul}}(\mathbb{R}^n)$ and $\partial_j \Phi(u) \in \mathcal{H}^{s-1}_{\mathtt{ul}}(\mathbb{R}^n)$, $j = 1, ..., n$ imply $\Phi(u) \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$. $\square$

Corollary 4.11 (Kato)**.** *Suppose that $s > \max\{n/2, 3/4\}$.*

(a) *If $u \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$ satisfies the condition*

$$|u(x)| \geq c > 0, \quad x \in \mathbb{R}^n,$$

then

$$\frac{1}{u} \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n).$$

(b) *If $u \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$, then $\overline{u(\mathbb{R}^n)}$ is the spectrum of the element u.*

Corollary 4.12. *Suppose that $s > \max\{n/2, 3/4\}$. If $u = (u_1, ..., u_d) \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)^d$, then*

$$\sigma_{\mathcal{H}^s_{\mathtt{ul}}}(u_1, ..., u_d) = \overline{u(\mathbb{R}^n)},$$

where $\sigma_{\mathcal{H}^s_{\mathtt{ul}}}(u_1, ..., u_d)$ is the joint spectrum of the elements $u_1, ..., u_d \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$.

Proof. Since

$$\delta_x : \mathcal{H}^{s'}_{\mathtt{ul}}(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n) \to \mathbb{C}, \quad w \to w(x),$$

is a multiplicative linear functional, Teorema 3.1.14 of [Hö2] implies the inclusion $\overline{u(\mathbb{R}^n)} \subset \sigma_{\mathcal{H}^s_{\mathtt{ul}}}(u_1, ..., u_d)$. On the other hand, if $\lambda = (\lambda_1, ..., \lambda_d) \notin \overline{u(\mathbb{R}^n)}$, then

$$u_\lambda = \overline{(u_1 - \lambda_1)}(u_1 - \lambda_1) + ... + \overline{(u_d - \lambda_d)}(u_d - \lambda_d) \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$$

satisfies the condition

$$u_\lambda(x) \geq c > 0, \quad x \in \mathbb{R}^n.$$

It follows that

$$\frac{1}{u_\lambda} \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$$

and

$$v_1(u_1 - \lambda_1) + ... + v_d(u_d - \lambda_d) = 1$$

with $v_1 = \overline{(u_1 - \lambda_1)}/u_\lambda, ..., v_d = \overline{(u_d - \lambda_d)}/u_\lambda \in \mathcal{H}^s_{\mathtt{ul}}(\mathbb{R}^n)$. The last equality expresses precisely that $\lambda \notin \sigma_{\mathcal{H}^s_{\mathtt{ul}}}(u_1, ..., u_d)$. $\square$

Corollary 4.13. *Suppose that* $s > \max\{n/2, 3/4\}$. *Let* $\Omega = \mathring{\Omega} \subset \mathbb{C}^d$ *and* $\Phi : \Omega \to \mathbb{C}$ *a holomorphic function.*

(a) *Let* $1 \leq p < \infty$. *If* $u = (u_1, ..., u_d) \in \mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)^d$ *satisfies the condition* $\overline{u_1(\mathbb{R}^n)} \times ... \times \overline{u_d(\mathbb{R}^n)} \subset \Omega$ *and if* $\Phi(0) = 0$, *then* $\Phi(u) \in \mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$.

(b) *If* $u = (u_1, ..., u_d) \in \mathcal{H}^s(\mathbb{R}^n)^d$ *satisfies the condition* $\overline{u_1(\mathbb{R}^n)} \times ... \times \overline{u_d(\mathbb{R}^n)} \subset \Omega$ *and if* $\Phi(0) = 0$, *then* $\Phi(u) \in \mathcal{H}^s(\mathbb{R}^n)$.

Proof. (a) Since $\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$ is an ideal in the algebra $\mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n)$, it follows that 0 belongs to the spectrum of any element of $\mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$. Hence $0 \in \overline{u_1(\mathbb{R}^n)} \times ... \times \overline{u_d(\mathbb{R}^n)} \subset \Omega$. Shrinking Ω if necessary, we can assume that $\Omega = \Omega_1 \times ... \times \Omega_d$ with $\overline{u_k(\mathbb{R}^n)} \subset \Omega_k$, $k = 1, ..., d$. Now we continue by induction on d.

Let $F : \Omega \to \mathbb{C}$ be the holomorphic function defined by

$$F(z_1, ..., z_d) = \begin{cases} \frac{\Phi(z_1,...,z_d) - \Phi(0,...,z_d)}{z_1} & if \quad z_1 \neq 0, \\ \frac{\partial \Phi}{\partial z_1}(0, ..., z_d) & if \quad z_1 = 0. \end{cases}$$

Then $\Phi(z_1, ..., z_d) = z_1 F(z_1, ..., z_d) + \Phi(0, ..., z_d)$, so

$$\Phi(u) = u_1 F(u) + \Phi(0, ..., u_d) \in \mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$$

because $u_1 F(u) \in \mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n) \cdot \mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n) \subset \mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$ and $\Phi(0, ..., u_d) \in \mathcal{K}_p^{\mathbf{s}}(\mathbb{R}^n)$ by inductive hypothesis.

(b) is a consequence of (a). □

Corollary 4.14 (A division lemma)**.** *Suppose that* $s > \max\{n/2, 3/4\}$. *Let* $t \in \mathbb{R}$ *such that* $s + t > n/2$. *Let* $u \in \mathcal{H}^t(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n)$ *and* $v \in \mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n)$. *If* v *satisfies the condition*

$$|v(x)| \geq c > 0, \quad x \in \mathtt{supp} u,$$

then

$$\frac{u}{v} \in \mathcal{H}^{\min\{s,t\}}(\mathbb{R}^n).$$

Proof. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $0 \leq \varphi \leq 1$, $\varphi = 1$ on $\mathtt{supp} u$, be such that

$$|v(x)| \geq c/2 > 0, \quad x \in \mathtt{supp} \varphi.$$

Then $w = \varphi |v|^2 + c^2(1 - \varphi)/4 \in \mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n)$ satisfies $w \geq (\varphi + (1 - \varphi)) c^2/4 = c^2/4$. If δ satisfies $0 < \delta < \min\{s + t - n/2, s - n/2\}$, then

$$\min\{s, t\} = \min\{s, t, s + t - n/2 - \delta\}.$$

By using Corollary 4.11 and Corollary 3.12 we obtain

$$\frac{u}{|v|^2} = \frac{u}{w} \in \mathcal{H}^t(\mathbb{R}^n) \cdot \mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n) \subset \mathcal{H}^{\min\{s,t\}}(\mathbb{R}^n)$$

This proves the lemma since

$$\frac{u}{v} = \overline{v} \cdot \frac{u}{|v|^2} \in \mathcal{H}_{\mathtt{ul}}^s(\mathbb{R}^n) \cdot \mathcal{H}^{\min\{s,t\}}(\mathbb{R}^n) \subset \mathcal{H}^{\min\{s,t\}}(\mathbb{R}^n).$$

□

References

[A-Ca] R. F. Arens and A. P. Calderdn, Analytic functions of several Banach algebra elements, *Ann. Math.*, **62** (1955), 2, 204-216.

[B1] A. Boulkhemair, Estimations L^2 précisées pour des intégrales oscillantes, *Comm. Partial Differential Equations,* **22**(1997), 1&2, 165-184.

[B2] A. Boulkhemair, Remarks on a Wiener type pseudodifferential algebra and Fourier integral operators, *Math.Res.Lett.,* **4**(1997), 53–67.

[CS] Thierry Coquand and Gabriel Stolzenberg, The Wiener lemma and certain of its generalizations, *Bull. Amer. Math. Soc. (N.S.)*, **24** (1991), 1, 1-9.

[DS] M. Dimassi, J. Sjostrand, *Spectral asymptotics in the semi-classical limit*, London Math. Soc. Lecture Note Series, 268. Cambridge University Press, Cambridge, 1999.

[Fe] H. Feichtinger, *Modulation spaces on locally compact abelian groups,* Technical report, University of Vienna, Vienna, 1983.

[Grö] K. Gröchenig, *Foundations of time frequency analysis*, Birkhäuser Boston Inc., Boston MA, 2001.

[Fo] G. B. Folland, *Real Analysis Modern Techniques and Their Applications*, Wiley, 1999.

[Hö1] L. Hörmander, *The analysis of linear partial differential operators*, I–IV, Grundlehren der math. Wissenschaften, 256-7, 274-5, Springer-Verlag (1983, 1985).

[Hö2] L. Hörmander, *An introduction to complex analysis in several variables*, North-Holland mathematical library, Amsterdam, New York, Oxford, Tokyo, 3rd ed., 1990.

[K] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Rational Mech. Anal.* **58** (1975), 3, 181–205.

[Co-Me] R. R. Coifman et Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Asterisque n°57, Soc. Math. France, Paris (1978).

[Ru1] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Company, New York, 1966.

[Ru2] W. Rudin, *Functional analysis*, McGraw-Hill Book Company, New York, 1973.

[S1] J. Sjöstrand, An algebra of pseudodifferential operators, *Math. Res. Lett.,* **1**(1994), 2, 189–192.

[S2] J. Sjöstrand, Wiener type algebras of pseudodifferential operators, *Séminaire EDP, École Polytechnique* (1994-95), Exposé 4.

[T1] J. Toft, Subalgebras to a Wiener type algebra of pseudo-differential operators, *Ann. Inst. Fourier*, **51**(2001), 1347-1383.

[Tri] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

Institute of Mathematics of The Romanian Academy, P.O. Box 1-174, RO-70700, Bucharest, Romania

E-mail address: Gruia.Arsu@imar.ro, agmilro@yahoo.com, agmilro@hotmail.com